

# Tilting generators via ample line bundles

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## Abstract

It is known that a tilting generator on an algebraic variety  $X$  gives a derived equivalence between  $X$  and a certain non-commutative algebra. In this paper, we present a method to construct a tilting generator from an ample line bundle, and construct it in several examples.

## 1 Introduction

Let  $D^b(X)$  be the bounded derived category of coherent sheaves on an algebraic variety  $X$ . Modern algebraic geometers have often observed that  $D^b(X)$  appears in a symmetry connecting two mathematical objects. For example, Beilinson [1] finds an example of such phenomena: he discovers that the derived category  $D^b(\mathbb{P}^n)$  on the projective space  $\mathbb{P}^n$  is equivalent to the derived category  $D^b(\text{mod End}_{\mathbb{P}^n}(\mathcal{E}))$  of the abelian category of finitely generated right  $\text{End}_{\mathbb{P}^n}(\mathcal{E})$ -modules, where  $\mathcal{E}$  is the vector bundle

$$\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-n).$$

We also have the so-called *McKay correspondence* ([4], [10]), which is a symmetry between complex algebraic geometry and representation theory. We now understand the McKay correspondence as a derived equivalence between an algebraic variety and a non-commutative algebra.

Van den Bergh proposes a generalization of Beilinson's theorem and the McKay correspondence through derived Morita theory [18].

**Theorem 1.1. [20, Theorem A]** *Let  $f: X \rightarrow Y = \text{Spec } R$  be a projective morphism between Noetherian schemes. Assume that  $f$  has at most one-dimensional fibers and  $\mathbb{R}f_*\mathcal{O}_X = \mathcal{O}_Y$ . Then there is a vector bundle  $\mathcal{E}$  on  $X$  such that the functor*

$$\mathbb{R}\text{Hom}_X(\mathcal{E}, -): D^b(X) \rightarrow D^b(\text{mod End}_X(\mathcal{E})),$$

*defines an equivalence of derived categories.*

Such a vector bundle  $\mathcal{E}$  is called a *tilting generator*. In the proof, Van den Bergh uses a globally generated ample line bundle  $\mathcal{L}$  on  $X$  and constructs  $\mathcal{E}$  from  $\mathcal{O}_X$  and  $\mathcal{L}^{-1}$ .

Recently, Kaledin [8] proved the existence of a tilting generator étale locally on  $Y$  when  $f: X \rightarrow Y$  is a crepant resolution and  $Y$  has symplectic singularities. He uses quite sophisticated tools such as mod  $p$  reductions and deformation quantizations, but it seems difficult to apply his method when  $Y$  does not have symplectic singularities.

The aim of this paper is to generalize Van den Bergh's arguments using ample line bundles, and to construct a tilting generator in a more general setting. In particular, we relax the fiber dimensionality assumption. One of our main results is:

**Theorem 1.2.** [Theorem 6.1] *Let  $f: X \rightarrow Y = \operatorname{Spec} R$  be a projective morphism between Noetherian schemes and  $R$  be a ring of finite type over a field, or a Noetherian complete local ring. Assume that  $f$  has at most two-dimensional fibers and  $\mathbb{R}f_*\mathcal{O}_X = \mathcal{O}_Y$ . Further assume that there is an ample globally generated line bundle  $\mathcal{L}$  on  $X$  that satisfies  $\mathbb{R}^2f_*\mathcal{L}^{-1} = 0$ . Then there is a tilting vector bundle generating the derived category  $D^-(X)$ .*

Our method can apply to more general situations: for instance, we can show that there is a tilting generator on  $X = T^*G(2, 4)$ , where  $G(2, 4)$  is the Grassmann manifold. The variety  $X$  admits the Springer resolution  $f: X \rightarrow \operatorname{Spec} R$ , which has a 4-dimensional fiber.

The paper is organized as follows. In §2, we show some easy results on ample line bundles, which we use later. In §3, we define tilting generators and explain their properties. In §4, we present our main construction of tilting generators and the assumptions behind it. In §5, we study the heart of a t-structure given in §4. The results in §5 are not used in any other sections. In §6, we prove Theorem 1.2 and find several examples where we can apply Theorem 1.2. In §7, we find a tilting generator of the derived category of the cotangent bundle of the Grassmann manifold  $G(2, 4)$ . In §8, we show an auxiliary result which is needed in §4.4. To prove the result in §8, we require the dualizing complex  $D_R$  on  $Y$  in Theorem 1.2. This requirement is why we assume that  $Y$  is a scheme of finite type over a field or a spectrum of a Noetherian complete local ring. In the appendix, we apply our result to prove the existence of non-commutative crepant resolutions in the sense of Van den Bergh ([19]).

**Notation and Conventions.** For a right (respectively, left) Noetherian (possibly non-commutative) ring  $A$ ,  $\operatorname{mod} A$  (respectively,  $A\operatorname{mod}$ ) is the abelian category of finitely generated right (respectively, left)  $A$ -modules and we set  $D^b(A) = D^b(\operatorname{mod} A)$ ,  $D^-(A) = D^-(\operatorname{mod} A)$  etc. We denote by  $A^\circ$  the opposite ring of a ring  $A$ .

For a Noetherian scheme  $X$ , we denote by  $D(X)$  (respectively,  $D^b(X)$ ,  $D^-(X)$ , ...) the unbounded (respectively, bounded, bounded above, ...) derived category of coherent sheaves. If  $\mathcal{A}$  is a sheaf of  $\mathcal{O}_X$ -algebras, then we

denote by  $\text{Coh } \mathcal{A}$  the category of right coherent  $\mathcal{A}$ -modules. Put  $D^-(\mathcal{A}) = D^-(\text{Coh } \mathcal{A})$ .

We also denote by  $D_X$  the dualizing complex (if it exists) and by  $\mathbb{D}_X$  the dualizing functor

$$\mathbb{R}\mathcal{H}om_X(-, D_X): D^-(X) \rightarrow D^+(X).$$

For a complex  $\mathcal{K}$  of coherent sheaves on  $X$ , we denote by  $\tau_{\leq p}\mathcal{K}(= \tau_{< p+1}\mathcal{K})$  and  $\tau_{> p}\mathcal{K}(= \tau_{\geq p+1}\mathcal{K})$  the following complexes:

$$(\tau_{\leq p}\mathcal{K})^n = \begin{cases} \mathcal{K}^n & n < p \\ \text{Ker } d^p & n = p \\ 0 & n > p \end{cases}$$

$$(\tau_{> p}\mathcal{K})^n = \begin{cases} 0 & n < p \\ \text{Im } d^p & n = p \\ \mathcal{K}^n & n > p. \end{cases}$$

Here,  $d^p: \mathcal{K}^p \rightarrow \mathcal{K}^{p+1}$  is the differential. Similarly we denote by  $\sigma_{\leq p}\mathcal{K}(= \sigma_{< p+1}\mathcal{K})$  and  $\sigma_{> p}\mathcal{K}(= \sigma_{\geq p+1}\mathcal{K})$  the following complexes:

$$(\sigma_{\leq p}\mathcal{K})^n = \begin{cases} \mathcal{K}^n & n \leq p \\ 0 & n > p \end{cases}$$

and

$$(\sigma_{> p}\mathcal{K})^n = \begin{cases} 0 & n \leq p \\ \mathcal{K}^n & n > p. \end{cases}$$

Then there are distinguished triangles in  $D(X)$ :

$$\tau_{\leq p}\mathcal{K} \rightarrow \mathcal{K} \rightarrow \tau_{> p}\mathcal{K} \rightarrow \tau_{\leq p}\mathcal{K}[1]$$

and

$$\sigma_{> p}\mathcal{K} \rightarrow \mathcal{K} \rightarrow \sigma_{\leq p}\mathcal{K} \rightarrow \sigma_{> p}\mathcal{K}[1].$$

We denote by  $D(X)^{\leq p}$  the full subcategory of  $D(X)$ :

$$D(X)^{\leq p} = \{\mathcal{K} \in D(X) \mid \mathcal{H}^i(\mathcal{K}) = 0 \text{ for all } i > p\}.$$

We also define  $D(A)^{\geq p} \dots$  similarly.

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## 2 Results on ample line bundles

In this section, we present some easy results on ample line bundles. Let  $f: X \rightarrow Y = \operatorname{Spec} R$  be a projective morphism from a Noetherian scheme to a Noetherian affine scheme. Suppose that  $\mathbb{R}^i f_* \mathcal{O}_X = 0$  for  $i > 0$  and the fibers of  $f$  are at most  $n$ -dimensional ( $n \geq 0$ ). Assume further that there is an ample, globally generated line bundle  $\mathcal{L}$  on  $X$ , satisfying

$$\mathbb{R}^i f_* \mathcal{L}^{-j} = 0 \quad (1)$$

for  $i \geq 2, 0 < j < n$ .

Take general elements  $H_k \in |\mathcal{L}|$ ,  $1 \leq k \leq n$ , and put  $H^k = H_1 \cap \cdots \cap H_k$ ,  $H^0 = X$  and  $H = H^1$ . Below we often use the following exact sequence:

$$0 \rightarrow \mathcal{L}^{l-1}|_{H^k} \rightarrow \mathcal{L}^l|_{H^k} \rightarrow \mathcal{L}^l|_{H^{k+1}} \rightarrow 0. \quad (2)$$

**Lemma 2.1.** *In the above situation, we have*

$$\mathbb{R}^i f_* \mathcal{L}^j = 0$$

for all  $i > 0, j \geq 0$ .

*Proof.* We show the assertion by induction on  $n$ , the upper bound of the dimension of the fibers of  $f$ . The statement obviously holds when  $f$  is quasi-finite, that is,  $n = 0$ . Next, suppose that  $n > 0$  and the statement holds for  $n - 1$ .

By (1) and (2), we see  $\mathbb{R}^1 f_*(\mathcal{O}_H) = 0$  and

$$\mathbb{R}^i f_*(\mathcal{L}^{-j}|_H) = 0$$

for  $i \geq 2, 0 \leq j < n - 1$ . Hence we can use the induction hypothesis, and conclude

$$\mathbb{R}^i f_*(\mathcal{L}^j|_H) = 0$$

for all  $i > 0, j \geq 0$ . Therefore, there is a surjection  $\mathbb{R}^i f_* \mathcal{L}^{j-1} \rightarrow \mathbb{R}^i f_* \mathcal{L}^j$ . Since  $\mathbb{R}^i f_* \mathcal{O}_X = 0$  for  $i > 0$ , we obtain the assertion.  $\square$

In the application below,  $X$  is always a smooth variety and  $-K_X$  is  $f$ -nef and  $f$ -big. If, furthermore,  $X$  is defined over  $\mathbb{C}$ , then Lemma 2.1 is automatically true by the vanishing theorem. (cf. [15, Theorem 1-2-5].) Next we see the following:

**Lemma 2.2.** *In the above situation, we have*

$$\mathbb{R}\operatorname{Hom}_X(\mathcal{L}^{-n}, C) \in R \operatorname{mod}$$

for  $C \in \operatorname{Coh} X$  with  $\mathbb{R}\operatorname{Hom}_X(\bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}, C) = 0$ , and

$$\mathbb{R}\operatorname{Hom}_X(\mathcal{L}^n, C) \in R \operatorname{mod}[-n]$$

for  $C \in \operatorname{Coh} X$  with  $\mathbb{R}\operatorname{Hom}_X(\bigoplus_{i=0}^{n-1} \mathcal{L}^i, C) = 0$ .

*Proof.* Take  $C \in \text{Coh } X$  such that

$$\mathbb{R}\Gamma(X, \bigoplus_{i=0}^{n-1} \mathcal{L}^i \otimes C) \cong \mathbb{R}\text{Hom}_X(\bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}, C) = 0.$$

Then we can show from (2) that  $\mathbb{R}\Gamma(H^k, \bigoplus_{i=k}^{n-1} \mathcal{L}^i \otimes C|_{H^k}) = 0$  for  $k = 0, \dots, n-1$  inductively. Therefore we have

$$\mathbb{R}\Gamma(X, \mathcal{L}^n \otimes C) \cong \mathbb{R}\Gamma(H, \mathcal{L}^n \otimes C|_H) \cong \dots \cong \mathbb{R}\Gamma(H^n, \mathcal{L}^n \otimes C|_{H^n}).$$

Because  $H^n$  is relative 0-dimensional, we obtain  $\mathbb{R}\text{Hom}_X(\mathcal{L}^{-n}, C) \in R \text{ mod}$  as required.

Take  $C \in \text{Coh } X$  such that

$$\mathbb{R}\Gamma(X, \bigoplus_{i=-n+1}^0 \mathcal{L}^i \otimes C) \cong \mathbb{R}\text{Hom}_X(\bigoplus_{i=0}^{n-1} \mathcal{L}^i, C) = 0.$$

Then we can show from (2) that  $\mathbb{R}\Gamma(H^k, \bigoplus_{i=-n+k+1}^0 \mathcal{L}^i \otimes C|_{H^k}) = 0$  for  $k = 0, \dots, n-1$  inductively. Therefore we have

$$\mathbb{R}\Gamma(X, \mathcal{L}^{-n} \otimes C) \cong \mathbb{R}\Gamma(H, \mathcal{L}^{-n+1} \otimes C|_H)[-1] \cong \dots \cong \mathbb{R}\Gamma(H^n, C|_{H^n})[-n].$$

Because  $H^n$  is 0-dimensional, we obtain  $\mathbb{R}\text{Hom}_X(\mathcal{L}^n, C) \in R \text{ mod}[-n]$   $\square$

The following lemma is fundamental in this paper.

**Lemma 2.3.** [20, Lemma 3.2.2] *Let  $f: X \rightarrow Y$  be a projective morphism between Noetherian schemes with at most  $n$ -dimensional fibers. Assume that  $Y$  is affine. Let  $\mathcal{L}$  be a globally generated ample line bundle on  $X$ . Then  $\bigoplus_{i=0}^n \mathcal{L}^i$  is a generator of  $D^-(X)$  (see the definition of generators in Definition 3.1.)*

### 3 Tilting generators

In this section, we define tilting generators on algebraic varieties.

Let  $f: X \rightarrow Y = \text{Spec } R$  be a projective morphism from a Noetherian scheme to an affine Noetherian scheme.

**Definition 3.1.** Let  $\mathcal{E}$  be a perfect complex on  $X$ : that is, locally  $\mathcal{E}$  is quasi-isomorphic to a bounded complex of finitely generated free  $\mathcal{O}_X$ -modules.

- (i)  $\mathcal{E}$  is said to be tilting if  $\text{Hom}_X^i(\mathcal{E}, \mathcal{E}) = 0$  for any  $i \neq 0$ .
- (ii)  $\mathcal{E}$  is called a generator of  $D^-(X)$  if the vanishing  $\mathbb{R}\text{Hom}_X(\mathcal{E}, \mathcal{K}) = 0$  for  $\mathcal{K} \in D^-(X)$  implies  $\mathcal{K} = 0$ .

**Example 3.2.** The vector bundle  $\mathcal{E} = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(-i)$  on  $\mathbb{P}^n$  is a tilting generator by Lemma 2.3. This fact was first observed by Beilinson [1].

For a tilting vector bundle  $\mathcal{E}$  on  $X$ , we denote by  $A$  the endomorphism algebra  $\text{End}_X(\mathcal{E})$  and define functors:

$$\begin{aligned}\Phi(-) &= \mathbb{R}\text{Hom}_X(\mathcal{E}, -): D^-(X) \longrightarrow D^-(A), \\ \Psi(-) &= - \overset{\mathbb{L}}{\otimes}_A \mathcal{E}: D^-(A) \longrightarrow D^-(X).\end{aligned}$$

Note that  $\Psi$  is a left adjoint functor of  $\Phi$  and  $\Phi \circ \Psi \cong \text{id}_{D^-(A)}$ .

The following lemma explains a characteristic property of tilting generators. The statement is well-known, but for the reader's convenience, we supply the proof.

**Lemma 3.3.** *In the above setting, assume furthermore that  $\mathcal{E}$  is a generator of  $D^-(X)$ . Then  $\Phi$  and  $\Psi$  define an equivalence of triangulated categories between  $D^-(X)$  and  $D^-(A)$ . This equivalence restricts to an equivalence between  $D^b(X)$  and  $D^b(A)$ .*

*Proof.* The isomorphism  $\Phi \circ \Psi \cong \text{id}_{D^-(A)}$  implies that the cone  $C$  of the adjunction morphism  $\Psi \circ \Phi(\mathcal{F}) \rightarrow \mathcal{F}$  for  $\mathcal{F} \in D^-(X)$  is annihilated by  $\Phi$ . Since  $\mathcal{E}$  is a generator of  $D^-(X)$ ,  $C$  is zero. In particular  $\Psi \circ \Phi \cong \text{id}_{D^-(X)}$ : that is,  $\Phi$  and  $\Psi$  define an equivalence of triangulated categories between  $D^-(X)$  and  $D^-(A)$ .

We can show that this equivalence restricts to an equivalence between  $D^b(X)$  and  $D^b(A)$ . It is obvious that  $\Phi(\mathcal{F}) \in D^b(A)$  for  $\mathcal{F} \in D^b(X)$ , so we only need to check that  $\Psi(M) \in D^b(X)$  for any  $M \in D^b(A)$ . To prove this fact, we may assume  $M \in \text{mod } A$ . For a sufficiently small integer  $m$ , consider the map

$$\phi: \tau_{<m} \Psi(M) \rightarrow \Psi(M)$$

induced by the canonical truncation  $\tau$ , and apply  $\Phi$  to it;

$$\Phi(\phi): \Phi(\tau_{<m} \Psi(M)) \rightarrow \Phi \circ \Psi(M) \cong M.$$

Then the map  $\Phi(\phi)$  is zero by the choice of  $m$ . Hence the map  $\phi$  is also zero, since  $\Phi: D^-(X) \rightarrow D^-(A)$  gives an equivalence. This implies  $\Psi(M) \in D^b(X)$ .  $\square$

## 4 Main construction

In this section, we show how to construct tilting generators from ample line bundles. The main result in this section is Theorem 4.16.

## 4.1 Setting

Let  $f: X \rightarrow Y = \operatorname{Spec} R$  be a projective morphism from a Noetherian scheme to an affine scheme of finite type over a field, or an affine scheme of a Noetherian complete local ring. Suppose that  $\mathbb{R}f_*\mathcal{O}_X = \mathcal{O}_Y$  and fibers of  $f$  are at most  $n$ -dimensional. Assume furthermore that there is an ample, globally generated line bundle  $\mathcal{L}$  on  $X$ , satisfying

$$\mathbb{R}^i f_* \mathcal{L}^{-j} = 0 \quad (3)$$

for  $i \geq 2, 0 < j < n$ . Then as shown in Lemma 2.1, we have

$$\mathbb{R}^i f_* \mathcal{L}^j = 0 \quad (4)$$

for all  $i > 0, j \geq 0$ . Furthermore, we know that  $\bigoplus_{i=0}^n \mathcal{L}^{-i}$  is a generator of  $D^-(X)$  by Lemma 2.3.

**Remark 4.1.** If we assume that (3) holds for  $i \geq 1$  and  $0 < j \leq n$ , then  $\bigoplus_{i=0}^n \mathcal{L}^{-i}$  is already a tilting generator, so there is nothing left to prove.

## 4.2 Orientation

For illustrative purposes, before explaining our construction, we sketch a proof of Theorem 1.1.

*Proof of Theorem 1.1.* First take the extension corresponding to a set of a generators of the  $R$ -module  $H^1(X, \mathcal{L}^{-1})$ ;

$$0 \rightarrow \mathcal{L}^{-1} \rightarrow \mathcal{N} \rightarrow \mathcal{O}_X^{\oplus r} \rightarrow 0. \quad (5)$$

Then by a direct calculation, we can show that  $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{N}$  is a tilting object. We can also see that  $\mathcal{E}$  is a generator of  $D^-(X)$  by Lemma 2.3.  $\square$

In the following subsections, we construct tilting vector bundles  $\mathcal{E}_k$  inductively as follows. First take  $\mathcal{E}_0 = \mathcal{O}_X$ , which is tilting by the assumption  $\mathbb{R}f_*\mathcal{O}_X = \mathcal{O}_Y$  (or (4)). Given a tilting vector bundle  $\mathcal{E}_{k-1}$  with  $0 < k \leq n-1$ , take the extension (9) as (5) and define a new tilting vector bundle  $\mathcal{E}_k$  as  $\mathcal{E}_{k-1} \oplus \mathcal{N}_{k-1}$ . To construct a tilting generator  $\mathcal{E}_n$ , we need a slightly more careful treatment, as explained in §4.4.

## 4.3 Inductive construction of tilting vector bundles

Under the setting in §4.1, we shall construct tilting vector bundles  $\mathcal{E}_k$  for  $0 \leq k \leq n-1$  inductively.

**Step 1.** Induction hypotheses.

Put  $\mathcal{E}_0 = \mathcal{O}_X$  and fix an integer  $k$  with  $0 < k \leq n-1$ . Assume that we have a tilting vector bundle  $\mathcal{E}_{k-1}$  on  $X$ . Let us denote the endomorphism algebra  $\text{End}_X \mathcal{E}_{k-1}$  by  $A_{k-1}$ . We also define the following functors:

$$\begin{aligned}\Phi_{k-1}(-) &= \mathbb{R}\text{Hom}_X(\mathcal{E}_{k-1}, -): D(X) \longrightarrow D(A_{k-1}), \\ \Psi_{k-1}(-) &= - \otimes_{A_{k-1}}^{\mathbb{L}} \mathcal{E}_{k-1}: D^-(A_{k-1}) \longrightarrow D^-(X).\end{aligned}$$

Note that  $\Phi_{k-1}$  restricts to the functor  $\Phi_{k-1}: D^-(X) \rightarrow D^-(A)$ , giving the right adjoint functor of  $\Psi_{k-1}$ .

As induction hypotheses, we assume the following.

- For any  $i \neq 0, 1$  and any  $l$  with  $0 < l \leq n-1$ , we have

$$\text{Hom}_X^i(\mathcal{E}_{k-1}, \mathcal{L}^{-l}) = 0. \quad (6)$$

- For any  $i \neq 0$  and any  $l$  with  $k-1 \leq l \leq n$ , we have

$$\text{Hom}_X^i(\mathcal{L}^{-l}, \mathcal{E}_{k-1}) = 0. \quad (7)$$

Note that if  $k=1$ , (6) and (7) hold by (3) and (4).

**Step 2.** Construction of  $\mathcal{E}_k$ .

Take a free  $A_{k-1}$  resolution of  $\Phi_{k-1}(\mathcal{L}^{-k})$  and denote it by  $P_{k-1}$ . Since  $\mathcal{H}^i(\Phi_{k-1}(\mathcal{L}^{-k})) = 0$  unless  $i = 0, 1$  by (6), we can take  $P_{k-1}$  satisfying  $P_{k-1}^i = 0$  for  $i \geq 2$ . We obtain a natural morphism  $\sigma_{\geq 1}(P_{k-1}) \rightarrow P_{k-1}$ , and hence we have a morphism  $\Psi_{k-1}(\sigma_{\geq 1}(P_{k-1})) \rightarrow \mathcal{L}^{-k}$ , since  $\Psi_{k-1}$  is a left adjoint functor of  $\Phi_{k-1}$ . Define an object  $\mathcal{N}_{k-1} \in D^-(X)$  to be the cone of this morphism;

$$\Psi_{k-1}(\sigma_{\geq 1}(P_{k-1})) \rightarrow \mathcal{L}^{-k} \rightarrow \mathcal{N}_{k-1} \rightarrow \Psi_{k-1}(\sigma_{\geq 1}(P_{k-1}))[1], \quad (8)$$

and we also define

$$\mathcal{E}_k = \mathcal{E}_{k-1} \oplus \mathcal{N}_{k-1}.$$

Applying  $\Phi_{k-1}$  to (8) and using the isomorphism,

$$\Phi_{k-1} \circ \Psi_{k-1} \cong \text{id}_{D^-(A_{k-1})},$$

we have  $\Phi_{k-1}(\mathcal{N}_{k-1}) \cong \sigma_{< 1}(P_{k-1})$ . Furthermore, we know that  $\Psi_{k-1}(\sigma_{\geq 1}(P_{k-1}))$  is isomorphic to an object of the form  $\mathcal{E}_{k-1}^{\oplus r_{k-1}}[-1]$  for some  $r_{k-1} \geq 0$ . Hence, there is a short exact sequence of coherent sheaves;

$$0 \rightarrow \mathcal{L}^{-k} \rightarrow \mathcal{N}_{k-1} \rightarrow \mathcal{E}_{k-1}^{\oplus r_{k-1}} \rightarrow 0. \quad (9)$$

Consequently,  $\mathcal{N}_{k-1}$  and  $\mathcal{E}_k$  are vector bundles on  $X$ .



**Step 3.**  $\mathcal{E}_k$  satisfies the induction hypotheses.

We shall check below that  $\mathcal{E}_k$  has similar properties to (6) and (7).

**Claim 4.2.**  $\text{Hom}_X^i(\mathcal{E}_k, \mathcal{L}^{-l}) = 0$  for any  $i \neq 0, 1$  and any  $l$  with  $0 < l \leq n-1$

*Proof.* Claim 4.2 follows from (3), (4), (6) and the long exact sequence

$$\rightarrow \text{Hom}_X^i(\mathcal{E}_{k-1}^{\oplus r}, \mathcal{L}^{-l}) \rightarrow \text{Hom}_X^i(\mathcal{N}_{k-1}, \mathcal{L}^{-l}) \rightarrow \text{Hom}_X^i(\mathcal{L}^{-k}, \mathcal{L}^{-l}) \rightarrow .$$

□

**Claim 4.3.**  $\text{Hom}_X^i(\mathcal{L}^{-l}, \mathcal{E}_k) = 0$  for any  $i \neq 0$  and any  $l$  with  $k \leq l \leq n$ .

*Proof.* Claim 4.3 follows from (4), (7) and the long exact sequence

$$\rightarrow \text{Hom}_X^i(\mathcal{L}^{-l}, \mathcal{L}^{-k}) \rightarrow \text{Hom}_X^i(\mathcal{L}^{-l}, \mathcal{N}_{k-1}) \rightarrow \text{Hom}_X^i(\mathcal{L}^{-l}, \mathcal{E}_{k-1}^{\oplus r}) \rightarrow .$$

□

**Claim 4.4.**  $\mathcal{E}_k$  is a tilting object.

*Proof.* From  $\Phi_{k-1}(\mathcal{N}_{k-1}) \cong \sigma_{<1}(P_{k-1})$ , we obtain

$$\text{Hom}_X^i(\mathcal{E}_{k-1}, \mathcal{N}_{k-1}) = \mathcal{H}^i(\Phi_{k-1}(\mathcal{N}_{k-1})) = 0 \quad (10)$$

for all  $i \neq 0$ . By (7) and the long exact sequence

$$\rightarrow \text{Hom}_X^i(\mathcal{E}_{k-1}^{\oplus r}, \mathcal{E}_{k-1}) \rightarrow \text{Hom}_X^i(\mathcal{N}_{k-1}, \mathcal{E}_{k-1}) \rightarrow \text{Hom}_X^i(\mathcal{L}^{-k}, \mathcal{E}_{k-1}) \rightarrow ,$$

we have

$$\text{Hom}_X^i(\mathcal{N}_{k-1}, \mathcal{E}_{k-1}) = 0 \quad (11)$$

for all  $i \neq 0$ . Finally, by Claim 4.3, (10) and the long exact sequence

$$\rightarrow \text{Hom}_X^i(\mathcal{E}_{k-1}^{\oplus r}, \mathcal{N}_{k-1}) \rightarrow \text{Hom}_X^i(\mathcal{N}_{k-1}, \mathcal{N}_{k-1}) \rightarrow \text{Hom}_X^i(\mathcal{L}^{-k}, \mathcal{N}_{k-1}) \rightarrow ,$$

we have

$$\text{Hom}_X^i(\mathcal{N}_{k-1}, \mathcal{N}_{k-1}) = 0 \quad (12)$$

for all  $i \neq 0$ . The equalities (10), (11) and (12) imply that  $\mathcal{E}_k$  is a tilting object. □

By induction on  $k$ , we can construct a tilting vector bundle  $\mathcal{E}_{n-1}$ .

**Remark 4.5.** We cannot apply our method in this subsection to construct  $\mathcal{E}_n$ . In Step 2, we need the vanishing of  $\text{Hom}_X^i(\mathcal{E}_{n-1}, \mathcal{L}^{-n})$  for  $i \geq 2$ . However this is not guaranteed by the induction hypothesis (6).

#### 4.4 Gluing t-structures

The vector bundle  $\mathcal{E}_{n-1}$  does not generate the category  $D^-(X)$  yet (see Lemma 4.6), so we need one more step to construct a tilting generator  $\mathcal{E}_n$ . As we mentioned in Remark 4.5, a similar method in §4.3 does not work. In this subsection, we make some assumptions and construct a tilting generator  $\mathcal{E}_n$  of  $D^-(X)$ .

As in §4.3, we define as  $A_{n-1} = \text{End}_X \mathcal{E}_{n-1}$  and

$$\begin{aligned}\Phi_{n-1}(-) &= \mathbb{R}\text{Hom}_X(\mathcal{E}_{n-1}, -): D(X) \longrightarrow D(A_{n-1}), \\ \Psi_{n-1}(-) &= - \otimes_{A_{n-1}}^{\mathbb{L}} \mathcal{E}_{n-1}: D^-(A_{n-1}) \longrightarrow D^-(X).\end{aligned}$$

Take a free  $A_{n-1}$  resolution  $P_{n-1}$  of  $\Phi_{n-1}(\mathcal{L}^{-n})$ . Then each  $\mathcal{H}^i(P_{n-1})$  vanishes for  $i < 0$  but does not necessarily vanish for  $i \geq 2$ . (cf. Remark 4.5.) As in Step 2 in §4.3, define an object  $\mathcal{N}_{n-1} \in D^b(X)$  such that  $\mathcal{N}_{n-1}$  fits into a triangle

$$\Psi_{n-1}(\sigma_{\geq 1}(P_{n-1})) \rightarrow \mathcal{L}^{-n} \rightarrow \mathcal{N}_{n-1} \rightarrow \Psi_{n-1}(\sigma_{\geq 1}(P_{n-1}))[1]. \quad (13)$$

Note that  $\mathcal{N}_{n-1}$  is a perfect complex, since so is  $\Psi_{n-1}(\sigma_{\geq 1}(P_{n-1}))$ . We again define

$$\mathcal{E}_n = \mathcal{E}_{n-1} \oplus \mathcal{N}_{n-1}.$$

Although we cannot conclude that  $\mathcal{E}_n$  is tilting, we consider the functor  $\Phi_n(-) = \mathbb{R}\text{Hom}_X(\mathcal{E}_n, -)$ .

Let us define  $\mathcal{C}_k$  for  $0 \leq k \leq n$  to be the full subcategory of the unbounded derived category  $D(X)$ ,

$$\mathcal{C}_k = \{\mathcal{K} \in D(X) \mid \Phi_k(\mathcal{K}) = 0\}.$$

**Lemma 4.6.** *Let  $k$  be an integer such that  $0 \leq k \leq n$ . Then  $\mathcal{K} \in \mathcal{C}_k$  if and only if*

$$\mathbb{R}\text{Hom}_X\left(\bigoplus_{i=0}^k \mathcal{L}^{-i}, \mathcal{K}\right) = 0.$$

*In particular,  $\mathcal{E}_n$  is a generator of  $D^-(X)$ .*

*Proof.* The proof proceeds by induction on  $k$ . First, note that the statement is true for  $k = 0$ , since  $\mathcal{O}_X = \mathcal{E}_0$ . For  $0 < k \leq n-1$ , we obtain from (9) that

$$\begin{aligned}\mathcal{K} \in \mathcal{C}_k &\iff \mathbb{R}\text{Hom}_X(\mathcal{E}_{k-1}, \mathcal{K}) \cong \mathbb{R}\text{Hom}_X(\mathcal{L}^{-k}, \mathcal{K}) = 0 \\ &\iff \mathcal{K} \in \mathcal{C}_{k-1} \text{ and } \mathbb{R}\text{Hom}_X(\mathcal{L}^{-k}, \mathcal{K}) = 0.\end{aligned}$$

For  $k = n$ , we have a similar conclusion by (13), since each term of the complex  $\Psi_{n-1}(\sigma_{\geq 1}(P_{n-1}))[1]$  is a direct sum of  $\mathcal{E}_{n-1}$ .

Suppose that  $\mathbb{R}\text{Hom}_X(\mathcal{E}_n, \mathcal{K}) = 0$  for  $\mathcal{K} \in D^-(X)$ . Then, the assertion we proved above and Lemma 2.3 imply that  $\mathcal{K} = 0$ , which implies the last statement.  $\square$

**Remark 4.7.** (i) In the setting in §4.1, assume furthermore

$$\mathbb{R}^i f_* \mathcal{L}^{-n} = 0 \quad (14)$$

for  $i \geq 2$ : that is, the vanishing in (3) for  $j = n$ . Then we can show that  $\mathcal{E}_n$  is a tilting vector bundle that generates  $D^-(X)$  as follows: In this case, we can show  $\mathrm{Hom}_X^i(\mathcal{E}_{n-1}, \mathcal{L}^{-n}) = 0$  for  $i \geq 2$  as Claim 4.3 and so the inductive construction in §4.3 works for  $\mathcal{E}_n$  (see Remark 4.5). By the lemma above,  $\mathcal{E}_n$  is a generator.

In particular, in this extra condition (14) for  $n = 2$ , our main Theorem 1.2 becomes rather obvious.

(ii) In (i), there is a short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{L}^{-k} \rightarrow \mathcal{N}_{k-1} \rightarrow \mathcal{E}_{k-1}^{\oplus r_{k-1}} \rightarrow 0 \quad (15)$$

for all  $k$  with  $1 \leq k \leq n$ , and some  $r_{k-1} \geq 0$ . Moreover we have

$$\mathcal{E}_n = \mathcal{O}_X \oplus \bigoplus_{k=0}^{n-1} \mathcal{N}_k.$$

We can easily see that the dual vector bundle  $\mathcal{E}^\vee$  of  $\mathcal{E}$  is also a tilting generator of  $D^-(X)$ .

Let us return to the situation in §4.1. Instead of assuming (14), we shall work under the following assumption until the end of this section.

**Assumption 4.8.** *For an object  $\mathcal{K} \in D(X)$ , if we have the equality*

$$\mathbb{R}\mathrm{Hom}_X\left(\bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}, \mathcal{K}\right) = 0,$$

*then the equality*

$$\mathbb{R}\mathrm{Hom}_X\left(\bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}, \mathcal{H}^k(\mathcal{K})\right) = 0$$

*holds for all  $k$ .*

In §6 and §7, we will study the cases where Assumption 4.8 holds. Assumption 4.8 means that  $\mathcal{K} \in \mathcal{C}_{n-1}$  implies  $\mathcal{H}^k(\mathcal{K}) \in \mathcal{C}_{n-1}$  for all  $k$ . Then we can define a t-structure on  $\mathcal{C}_{n-1}$  induced by the standard one on  $D(X)$ . Next we introduce the triangulated category

$$D^\dagger(X) = \left\{ \mathcal{K} \in D(X) \mid \Phi_{n-1}(\mathcal{K}) \in D^b(A_{n-1}) \right\}.$$

Note that  $\Psi_{n-1}$  defines a functor from  $D^b(A_{n-1})$  to  $D^\dagger(X)$ , since  $\Phi_{n-1} \circ \Psi_{n-1} \cong \mathrm{id}_{D^b(A_{n-1})}$ . Hence it is a left adjoint functor of  $\Phi_{n-1}: D^\dagger(X) \rightarrow$

$D^b(A_{n-1})$ . The advantage of considering  $D^\dagger(X)$  is the existence of a right adjoint functor

$$\Psi'_{n-1}: D^b(A_{n-1}) \rightarrow D^\dagger(X)$$

of  $\Phi_{n-1}: D^\dagger(X) \rightarrow D^b(A_{n-1})$  (see Lemma 8.2). Therefore, we can construct a new t-structure on  $D^\dagger(X)$  by gluing the t-structure on  $\mathcal{C}_{n-1}$  (with perversity  $p \in \mathbb{Z}$ ) and the standard t-structure on  $D^b(A_{n-1})$  via the exact triple of triangulated categories [6, pp. 286];

$$\mathcal{C}_{n-1} \xrightarrow{i_{n-1}^*} D^\dagger(X) \rightarrow D^b(A_{n-1}).$$

Let  $i_{n-1}^*, i_{n-1}^!: D^\dagger(X) \rightarrow \mathcal{C}_{n-1}$  be the left and right adjoint functors of the inclusion functor  $i_{n-1}$  respectively, whose existence follows from the existence of the right and left adjoint functors of  $\Phi_{n-1}$ . Specifically,  $(i_{n-1}^*, i_{n-1}^!)$  are constructed so that there are distinguished triangles

$$\begin{aligned} \Psi_{n-1} \circ \Phi_{n-1}(E) &\rightarrow E \rightarrow i_{n-1}^* E, \\ i_{n-1}^!(E) &\rightarrow E \rightarrow \Psi'_{n-1} \circ \Phi_{n-1}(E) \end{aligned}$$

for any  $E \in D^\dagger(X)$ . We, therefore, obtain the new t-structure on  $D^\dagger(X)$ :

$$\begin{aligned} {}^p\mathcal{D}^{\leq 0} &= \{\mathcal{K} \in D^\dagger(X) \mid \Phi_{n-1}(\mathcal{K}) \in D^b(A_{n-1})^{\leq 0}, i_{n-1}^* \mathcal{K} \in \mathcal{C}_{n-1}^{\leq p}\}, \\ {}^p\mathcal{D}^{\geq 0} &= \{\mathcal{K} \in D^\dagger(X) \mid \Phi_{n-1}(\mathcal{K}) \in D^b(A_{n-1})^{\geq 0}, i_{n-1}^! \mathcal{K} \in \mathcal{C}_{n-1}^{\geq p}\}. \end{aligned}$$

Here,  $p$  is an integer that determines the *perversity* of the t-structure and we denote  $\mathcal{C}_{n-1}^{\leq p} = \mathcal{C}_{n-1} \cap D(X)^{\leq p}$  and  $\mathcal{C}_{n-1}^{\geq p} = \mathcal{C}_{n-1} \cap D(X)^{\geq p}$ . The heart of the above t-structure is called the category of *perverse coherent sheaves* (cf. [3]):

$${}^p\text{Per}(X/A_{n-1}) = \left\{ \mathcal{K} \in D^\dagger(X) \left| \begin{array}{l} \Phi_{n-1}(\mathcal{K}) \in \text{mod } A_{n-1} \text{ and} \\ i_{n-1}^* \mathcal{K} \in \mathcal{C}_{n-1}^{\leq p}, i_{n-1}^! \mathcal{K} \in \mathcal{C}_{n-1}^{\geq p} \end{array} \right. \right\}.$$

**Remark 4.9.** Note that since the functor  $\Phi_{n-1}: D^b(X) \rightarrow D^b(A_{n-1})$  does not necessarily have a right adjoint functor, we cannot construct the perverse t-structure on  $D^b(X)$  in a similar way. However we will see in §5 that  ${}^0\text{Per}(X/A_{n-1})$  is in fact the heart of a bounded t-structure on  $D^b(X)$ .

**Remark 4.10.** The condition  $i_{n-1}^* \mathcal{K} \in \mathcal{C}_{n-1}^{\leq p}$  (resp.  $i_{n-1}^! \mathcal{K} \in \mathcal{C}_{n-1}^{\geq p}$ ) is equivalent to the condition

$$\text{Hom}_X(\mathcal{K}, C) = 0 \quad (\text{resp. } \text{Hom}_X(C, \mathcal{K}) = 0) \tag{16}$$

for any  $C \in \mathcal{C}_{n-1}^{\geq p+1}$  (resp.  $C \in \mathcal{C}_{n-1}^{\leq p-1}$ ). If  $\mathcal{K} \in D^b(X)$ , it is enough to check (16) for  $C \in \mathcal{C}_{n-1} \cap \text{Coh } X[j]$  with  $j < -p$  (resp.  $j > -p$ ).

**Claim 4.11.** *The object  $\mathcal{N}_{n-1}$  belongs to  ${}^0\text{Per}(X/A_{n-1})$ .*

*Proof.* Since  $\mathcal{N}_{n-1} \in D^b(X)$ , it is enough to check the following;

$$\Phi_{n-1}(\mathcal{N}_{n-1}) \in \text{mod } A_{n-1}, \quad (17)$$

$$\text{Hom}_X^i(\mathcal{N}_{n-1}, C) = 0 \text{ for } i < 0 \text{ and } C \in \mathcal{C}_{n-1} \cap \text{Coh } X, \quad (18)$$

$$\text{Hom}_X^i(C, \mathcal{N}_{n-1}) = 0 \text{ for } i < 0 \text{ and } C \in \mathcal{C}_{n-1} \cap \text{Coh } X. \quad (19)$$

First let us check (17). By the triangle (13), we have  $\Phi_{n-1}(\mathcal{N}_{n-1}) \cong \sigma_{\leq 0} P_{n-1}$ , hence  $\mathcal{H}^i(\Phi_{n-1}(\mathcal{N}_{n-1})) = 0$  for  $i > 0$ . For  $i < 0$ , we have

$$\begin{aligned} \mathcal{H}^i(\Phi_{n-1}(\mathcal{N}_{n-1})) &\cong \mathcal{H}^i(\Phi_{n-1}(\mathcal{L}^{-n})) \\ &= 0, \end{aligned}$$

since  $\mathcal{E}_{n-1}$  and  $\mathcal{L}^{-n}$  are vector bundles on  $X$ . Therefore (17) holds. Next for  $C \in \mathcal{C}_{n-1} \cap \text{Coh } X$ , we have

$$\text{Hom}_X^i(\mathcal{N}_{n-1}, C) \cong \text{Hom}_X^i(\mathcal{L}^{-n}, C) \quad (20)$$

for any  $i$  by the triangle (13). Therefore (18) follows. Finally we check (19). Since  $\mathcal{L}^{-n}$  and  $\Psi_{n-1}(\sigma_{\geq 1}(P_{n-1})[1])$  belong to  $D(X)^{\geq 0}$ , we have

$$\begin{aligned} \text{Hom}_X^i(C, \mathcal{L}^{-n}) &\cong \text{Hom}_X^i(C, \Psi_{n-1}(\sigma_{\geq 1}(P_{n-1})[1])) \\ &= 0 \end{aligned}$$

for  $i < 0$  and  $C \in \mathcal{C}_{n-1} \cap \text{Coh } X$ . By the triangle (13), (19) also follows.  $\square$

**Claim 4.12.** *For  $i > 0$  and  $B \in {}^0\text{Per}(X/A_{n-1})$ , we have*

$$\text{Hom}_X^i(\mathcal{N}_{n-1}, B) = 0. \quad (21)$$

*In particular,  $\mathcal{N}_{n-1}$  is a projective object of  ${}^0\text{Per}(X/A_{n-1})$ .*

*Proof.* We have a triangle

$$\Psi_{n-1} \circ \Phi_{n-1}(B) \rightarrow B \rightarrow i_{n-1}^* B. \quad (22)$$

By the definition of  ${}^0\text{Per}(X/A_{n-1})$ , we have  $i_{n-1}^* B \in \mathcal{C}_{n-1}^{\leq 0}$ . To see (21), it suffices to show

$$\text{Hom}_X^i(\mathcal{N}_{n-1}, i_{n-1}^* B) = 0 \quad (23)$$

and

$$\text{Hom}_X^i(\mathcal{N}_{n-1}, \Psi_{n-1} \circ \Phi_{n-1}(B)) = 0 \quad (24)$$

for  $i > 0$ . To prove (23), it is enough to show

$$\text{Hom}_X^i(\mathcal{N}_{n-1}, C) = 0$$

for any  $i > 0$  and  $C \in \mathcal{C}_{n-1} \cap \text{Coh } X$ . Then the assertion follows from (20) and Lemma 2.2.

Next let us show (24). By the triangle (13), it is enough to check the following;

$$\text{Hom}_X^i(\Psi_{n-1}(\sigma_{\geq 1}(P_{n-1})[1]), \Psi_{n-1} \circ \Phi_{n-1}(B)) = 0, \quad (25)$$

$$\text{Hom}_X^i(\mathcal{L}^{-n}, \Psi_{n-1} \circ \Phi_{n-1}(B)) = 0 \quad (26)$$

for  $i > 0$ . Note that

$$(25) \cong \text{Hom}_{A_{n-1}}^i(\sigma_{\geq 1}(P_{n-1})[1], \Phi_{n-1}(B)). \quad (27)$$

Since  $\Phi_{n-1}(B) \in \text{mod } A_{n-1}$ ,  $\sigma_{\geq 1}(P_{n-1})[1] \in D^b(A_{n-1})^{\geq 0}$  and each term of  $\sigma_{\geq 1}(P_{n-1})[1]$  is a projective  $A_{n-1}$ -module, we conclude (27) = 0. In order to check (26), let us take a free  $A_{n-1}$  resolution  $\mathcal{Q} = (\cdots \rightarrow \mathcal{Q}^{-1} \rightarrow \mathcal{Q}^0 \rightarrow 0)$  of an  $A_{n-1}$ -module  $\Phi_{n-1}(B)$ . Then each term of  $\Psi_{n-1}(\mathcal{Q})$  is a direct sum of  $\mathcal{E}_{n-1}$ . Hence by Claim 4.3, we conclude (26) holds.  $\square$

We readily see that  $\mathcal{E}_{n-1} \in {}^0\text{Per}(X/A_{n-1})$ , and therefore we have  $\mathcal{E}_n \in {}^0\text{Per}(X/A_{n-1})$ .

**Claim 4.13.**  $\mathcal{E}_n$  is a tilting object.

*Proof.* (17) yields

$$\text{Hom}_X^i(\mathcal{E}_{n-1}, \mathcal{N}_{n-1}) = 0$$

for all  $i \neq 0$ . Also Claim 4.12 implies that

$$\begin{aligned} \text{Hom}_X^i(\mathcal{N}_{n-1}, \mathcal{N}_{n-1}) &\cong \text{Hom}_X^i(\mathcal{N}_{n-1}, \mathcal{E}_{n-1}) \\ &= 0 \end{aligned}$$

for all  $i \neq 0$ . Moreover recalling that  $\mathcal{E}_{n-1}$  is a tilting vector bundle, we have  $\text{Hom}_X^i(\mathcal{E}_{n-1}, \mathcal{E}_{n-1})$  vanishes for  $i \neq 0$ . Combining these equalities, we see that  $\mathcal{E}_n = \mathcal{E}_{n-1} \oplus \mathcal{N}_{n-1}$  is a tilting object in  $D^b(X)$ .  $\square$

**Claim 4.14.**  $\mathcal{E}_n$  is a vector bundle.

*Proof.* It is enough to show that  $\mathcal{N}_{n-1}$  is a vector bundle. By Lemma 4.15, we know  $\Phi_{n-1}(\mathcal{O}_x) \in \text{mod } A_{n-1}$  for any closed points  $x \in X$ , which implies  $\mathcal{O}_x \in {}^0\text{Per}(X/A_{n-1})$ . Hence it follows from Claim 4.12 that  $\mathbb{R}\text{Hom}_X(\mathcal{N}_{n-1}, \mathcal{O}_x) \in R\text{mod}$ , and in particular  $\mathcal{N}_{n-1}$  is a vector bundle by Lemma 4.15.  $\square$

**Lemma 4.15.** [2, Lemma 4.3] *For a Noetherian scheme  $X$  and an object  $\mathcal{E} \in D^b(X)$ , the following are equivalent.*

(i)  $\mathcal{E}$  is a vector bundle.

(ii)  $\mathrm{Hom}_X^i(\mathcal{E}, \mathcal{O}_x) = 0$  for any points  $x \in X$  and  $i \neq 0$ .

Combining the above argument and Lemma 4.6, we can prove the following theorem.

**Theorem 4.16.** *Let  $f$  and  $\mathcal{L}$  be as in §4.1. Assume that Assumption 4.8 is satisfied. Then there is a vector bundle  $\mathcal{E}$  such that  $\mathcal{E}$  is a tilting generator of  $D^-(X)$ .*

**Remark 4.17.** For  $n = 2$ , we can show that Assumption 4.8 is always satisfied in §6.1. Since we have proved that  $\mathcal{N}_1$  is a vector bundle, taking the cohomology of (13) yields  $\mathrm{Ext}_X^2(\mathcal{E}_1, \mathcal{L}^{-2}) \otimes_{A_1} \mathcal{E}_1 = 0$ . As  $\mathrm{Ext}_X^2(\mathcal{E}_1, \mathcal{L}^{-2})$  may be non-zero, this vanishing is not obvious.

## 5 The hearts of t-structures

Let  $f$  and  $\mathcal{L}$  be as in §4.1 and furthermore assume that Assumption 4.8 holds. Below we use the same notation as in §4, but we omit the index  $n$ , for instance  $\mathcal{E} = \mathcal{E}_n$ ,  $A = A_n = \mathrm{End}_X(\mathcal{E}_n)$ , etc. Recall that the equivalence

$$\Phi = \mathbb{R}\mathrm{Hom}_X(\mathcal{E}, -): D^-(X) \longrightarrow D^-(A)$$

induces an equivalence between  $D^b(X)$  and  $D^b(A)$  by Lemma 3.3.

The aim of this section is the following, which will not be used in any subsequent sections. Recall that  ${}^0\mathrm{Per}(X/A_{n-1})$  is, by definition, the heart of the t-structure  $({}^0\mathcal{D}^{\leq 0}, {}^0\mathcal{D}^{\geq 0})$  on  $D^\dagger(X)$ .

**Proposition 5.1.** *The abelian category  ${}^0\mathrm{Per}(X/A_{n-1})$  is the heart of a bounded t-structure on  $D^b(X)$ , and  $\Phi({}^0\mathrm{Per}(X/A_{n-1})) = \mathrm{mod} A$ .*

*Proof.* We first show that  ${}^0\mathrm{Per}(X/A_{n-1}) \subset D^b(X)$ . For an object  $E \in {}^0\mathcal{D}^{\leq 0}$ , we have the distinguished triangle in  $D^\dagger(X)$

$$\Psi_{n-1} \circ \Phi_{n-1}(E) \rightarrow E \rightarrow i_{n-1}^*(E). \quad (28)$$

By the definition of  ${}^0\mathcal{D}^{\leq 0}$ , we have  $\Phi_{n-1}(E) \in D^b(A_{n-1})^{\leq 0}$  and  $i_{n-1}^*(E) \in \mathcal{C}_{n-1}^{\leq 0}$ . Therefore  $\Psi_{n-1} \circ \Phi_{n-1}(E)$  and  $i_{n-1}^*(E)$  are objects in  $D(X)^{\leq 0}$ , hence (28) yields  $E \in D(X)^{\leq 0}$ . In particular, we have  ${}^0\mathrm{Per}(X/A_{n-1}) \subset D^-(X)$ . On the other hand, Claim 4.12 implies that the equivalence  $\Phi: D^-(X) \rightarrow D^-(A)$  takes  ${}^0\mathrm{Per}(X/A_{n-1})$  to  $\mathrm{mod} A$ . Since  $\Phi$  restricts to an equivalence between  $D^b(X)$  and  $D^b(A)$ , we must have  ${}^0\mathrm{Per}(X/A_{n-1}) \subset D^b(X)$ .

Let  $(\tau_{\leq 0}^0, \tau_{\geq 0}^0)$  be the truncation functors corresponding to the t-structure  $({}^0\mathcal{D}^{\leq 0}, {}^0\mathcal{D}^{\geq 0})$ . In order to conclude that  ${}^0\mathrm{Per}(X/A_{n-1})$  is the heart of a bounded t-structure of  $D^b(X)$ , it is enough to show that for any object  $E \in D^b(X)$ , we have  $\tau_{\leq -i}^0(E) = \tau_{\geq i}^0(E) = 0$  for  $i \gg 0$ .

Since the functor  $\Phi_{n-1}: D^\dagger(X) \rightarrow D^b(A_{n-1})$  takes  $({}^0\mathcal{D}^{\leq 0}, {}^0\mathcal{D}^{\geq 0})$  to  $(D^b(A_{n-1})^{\leq 0}, D^b(A_{n-1})^{\geq 0})$ , we have

$$\Phi_{n-1}(\tau_{\geq i}^0(E)) \cong \tau_{\geq i}^A(\Phi_{n-1}(E)), \quad (29)$$

where  $(\tau_{\leq 0}^A, \tau_{\geq 0}^A)$  are the truncation functors with respect to the standard t-structure on  $D^b(A_{n-1})$ . Since  $\Phi_{n-1}(E) \in D^b(A_{n-1})$ , we have  $(29) = 0$  for  $i \gg 0$ . Therefore  $\tau_{\geq i}^0(E) \in \mathcal{C}_{n-1}^{\geq i} \subset D(X)^{\geq i}$ . On the other hand, since  $E \in D^b(X)$ , we have  $\text{Hom}(E, F) = 0$  for  $F \in D(X)^{\geq i}$  for  $i \gg 0$ . Therefore the natural morphism  $E \rightarrow \tau_{\geq i}^0(E)$  is zero, which implies  $\tau_{\geq i}^0(E) = 0$  for  $i \gg 0$ . By a similar argument, we have  $\tau_{\leq -i}^0(E) = 0$  for  $i \gg 0$ .

Since both of  $\Phi({}^0\text{Per}(X/A_{n-1}))$  and  $\text{mod } A$  are the hearts of bounded t-structures on  $D^b(A)$ , and we also know  $\Phi({}^0\text{Per}(X/A_{n-1})) \subset \text{mod } A$ , we obtain

$$\Phi({}^0\text{Per}(X/A_{n-1})) = \text{mod } A.$$

□

Assume furthermore that the equality (14) holds. Then Remark 4.7 implies that  $\mathcal{E}$  and  $\mathcal{E}^\vee$  are tilting generators of  $D^-(X)$ . We define the functor

$$\Phi_k^\vee = \mathbb{R}\text{Hom}_X(\mathcal{E}_k^\vee, -): D(X) \longrightarrow D(A_k^\circ),$$

and then  $\Phi^\vee = \Phi_n^\vee$  gives an equivalence between  $D^b(X)$  and  $D^b(A^\circ)$ . Here, we identify  $D(A_k^\circ)$  with  $D(\text{End}_X(\mathcal{E}_k^\vee))$ , using the isomorphism  $A_k^\circ \cong \text{End}_X(\mathcal{E}_k^\vee)$ .

Define the full subcategories of the unbounded derived category  $D(X)$  as

$$\begin{aligned} \mathcal{C}_{n-1}^\vee &= \{\mathcal{K} \in D(X) \mid \Phi_{n-1}^\vee(\mathcal{K}) = 0\} \\ D^{\dagger\dagger}(X) &= \left\{ \mathcal{K} \in D(X) \mid \Phi_{n-1}^\vee(\mathcal{K}) \in D^b(A_{n-1}^\circ) \right\}. \end{aligned}$$

It is easy to see from Lemma 4.6 that for an object  $\mathcal{K} \in D(X)$ ,  $\mathcal{K}$  belongs to  $\mathcal{C}_{n-1}^\vee$  if and only if  $\mathcal{K} \otimes \mathcal{L}^{\otimes -n+1}$  belongs to  $\mathcal{C}_{n-1}$ . Therefore we can check that  $\mathcal{K} \in \mathcal{C}_{n-1}^\vee$  implies that  $\mathcal{H}^k(\mathcal{K}) \in \mathcal{C}_{n-1}^\vee$  for all  $k$  by Assumption 4.8, and hence by the exact triple of triangulated categories

$$\mathcal{C}_{n-1}^\vee \rightarrow D^{\dagger\dagger}(X) \xrightarrow{\Phi_{n-1}^\vee} D^b(A_{n-1}^\circ),$$

we can define the category of perverse coherent sheaves  ${}^p\text{Per}(X/A_{n-1}^\circ)$  as  ${}^p\text{Per}(X/A_{n-1})$ .

Note that (15) yields  $\mathbb{R}\text{Hom}_X(\mathcal{N}_{n-1}^\vee, C) = \mathbb{R}\text{Hom}_X(\mathcal{L}^n, C)$  for  $C \in \mathcal{C}_{n-1}^\vee$ , hence Lemma 2.2 implies

$$\mathbb{R}\text{Hom}_X(\mathcal{N}_{n-1}^\vee, C) \in R\text{mod}$$



for  $C \in \mathcal{C}_{n-1}^\vee \cap \text{Coh } X[n]$ . In particular, we see

$$\Phi^\vee(\mathcal{C}_{n-1}^\vee \cap \text{Coh } X[n]) \subset \text{mod } A^\circ, \quad (30)$$

The proof of the next proposition uses this fact. (By comparison,

$$\Phi(\mathcal{C}_{n-1} \cap \text{Coh } X) \subset \text{mod } A$$

holds by Lemma 2.2 and (20). The proof of Claims 4.11 and 4.12 relies on this fact.)

**Proposition 5.2.** *In the setting of Proposition 5.1, assume furthermore the equality (14) holds. Then the abelian category  ${}^{-n}\text{Per}(X/A_{n-1}^\circ)$  is the heart of a bounded  $t$ -structure on  $D^b(X)$ , and  $\Phi^\vee({}^{-n}\text{Per}(X/A_{n-1}^\circ)) = \text{mod } A^\circ$ .*

*Proof.* We outline the proof and leave the details to the reader. First we show that the object  $\mathcal{N}_{n-1}^\vee$  belongs to  ${}^{-n}\text{Per}(X/A_{n-1}^\circ)$  as Claim 4.11. In the proof, we use (30).

Next, we mimic the proof of Claim 4.12 and show

$$\text{Hom}_X^i(\mathcal{N}_{n-1}^\vee, B) = 0$$

for  $i > 0$  and  $B \in {}^{-n}\text{Per}(X/A_{n-1}^\circ)$ . We again use (30) here.

From these facts, we can conclude

$$\Phi^\vee({}^{-n}\text{Per}(X/A_{n-1}^\circ)) \subset \text{mod } A^\circ$$

and then a similar argument to Proposition 5.1 works.  $\square$

**Example 5.3.** In this example, we show that tilting generators induce the derived equivalence between certain varieties connected by a Mukai flop. We also apply Propositions 5.1 and 5.2.

Let  $X$  be the cotangent bundle  $T^*\mathbb{P}^n$  of the projective space  $\mathbb{P}^n$  ( $n \geq 2$ ) and  $g: Z \rightarrow X$  a blow-up along the zero section of the projection  $\pi: X \rightarrow \mathbb{P}^n$ . The exceptional locus  $E(\subset Z)$  of  $g$  is the incidence variety in  $\mathbb{P}^n \times (\mathbb{P}^n)^\vee$ , where  $(\mathbb{P}^n)^\vee$  is the dual projective space. By contracting curves contained in fibers of the projection  $E \rightarrow (\mathbb{P}^n)^\vee$ , we obtain a birational contraction  $g^+: Z \rightarrow X^+ = T^*((\mathbb{P}^n)^\vee)$ . The resulting birational map

$$\phi = g^+ \circ g^{-1}: X \dashrightarrow X^+$$

is so called a *Mukai flop*. Put  $R = \text{Spec } H^0(X, \mathcal{O}_X)$ . Then we have a birational contraction

$$f: X \rightarrow Y = \text{Spec } R$$

which contracts only the zero section of  $\pi$ . In particular,  $f$  has at most  $n$ -dimensional fibers.

We put  $\mathcal{O}_X(1) = \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$ . Then by direct calculations (refer to calculations in §7) and Lemma 2.3 we know that  $\mathcal{E} = \bigoplus_{i=0}^n \mathcal{O}_X(-i)$  is a tilting generator of  $D^-(X)$ . On the other hand, we can see that (14) holds for  $\mathcal{L} = \mathcal{O}_X(1)$ . Apply the arguments in §4.3 and Remark 4.7; we obtain tilting vector bundles  $\mathcal{E}_k = \bigoplus_{i=0}^k \mathcal{O}_X(-i)$  for all  $k$  with  $0 \leq k \leq n$  (in other words,  $r_k = 0$  in (15) for all  $k$  with  $0 < k \leq n$ ). We can also check that Assumption 4.8 holds. Therefore we can apply Propositions 5.1 and 5.2.

In what follows, we use the same notation as in the previous section, and we also use the superscript  $+$  to denote the corresponding object on  $X^+$  to the object on  $X$ . For instance,  $\mathcal{E}^+ = \bigoplus_{i=0}^n \mathcal{O}_{X^+}(-i)$ .

Since  $\phi$  is isomorphic in codimension one, there is an equivalence between categories of reflexive sheaves on  $X$  and  $X^+$ . Hence, we have a reflexive sheaf  $\mathcal{E}'$  on  $X^+$  corresponding to  $\mathcal{E}$ , satisfying  $\text{End}_X(\mathcal{E}) \cong \text{End}_{X^+}(\mathcal{E}')$ . It is known that the corresponding reflexive sheaf on  $X^+$  to  $\mathcal{O}_X(-1)$  is  $\mathcal{O}_{X^+}(1)$  (cf. [16, Lemma 1.3], [17, Lemma 2.3.1]). From these facts, we see that  $\mathcal{E}' \cong (\mathcal{E}^+)^{\vee}$  and so we have an isomorphism of rings, denoted by  $\phi_*$ :

$$\phi_*: A = \text{End}_X(\mathcal{E}) \cong \text{End}_X(\mathcal{E}') \cong \text{End}_X((\mathcal{E}^+)^{\vee}) \cong A^{\circ}.$$

In particular, we have an equivalence  $D^b(A) \cong D^b(A^{\circ})$  preserving the hearts of the standard t-structures. Compose this equivalence with equivalences given by tilting generators  $\mathcal{E}$  and  $(\mathcal{E}^+)^{\vee}$ , and then we obtain an equivalence

$$\Xi: D^b(X) \rightarrow D^b(A) \rightarrow D^b(A^{\circ}) \rightarrow D^b(X^+),$$

which satisfies  $\Xi({}^0\text{Per}(X/A_{n-1})) = {}^{-n}\text{Per}(X^+/A_{n-1}^{\circ})$  by Propositions 5.1 and 5.2. Compare the results in [16] and [13, Corollary 5.7], where a similar derived equivalence is shown to exist by a very different method from ours.

## 6 The case of two-dimensional fibers

### 6.1 Main result

Let  $f: X \rightarrow Y = \text{Spec } R$  be a projective morphism from a Noetherian scheme to an affine scheme of finite type over a field, or an affine scheme of a Noetherian complete local ring. Suppose that the fibers of  $f$  are at most two-dimensional. Assume furthermore that  $\mathbb{R}f_*\mathcal{O}_X = \mathcal{O}_Y$  and there is an ample, globally generated line bundle  $\mathcal{L}$  on  $X$ , satisfying  $\mathbb{R}^2f_*\mathcal{L}^{-1} = 0$ . The following is a main theorem in this paper.

**Theorem 6.1.** *Under the above situation, there is a tilting vector bundle generating the derived category  $D^-(X)$ .*

*Proof.* We have to show that Assumption 4.8 holds so that we apply Theorem 4.16. Take  $\mathcal{K} \in D(X)$ , which satisfies

$$\mathbb{R}f_*\mathcal{K} = \mathbb{R}f_*(\mathcal{K} \otimes \mathcal{L}) = 0. \quad (31)$$

Let  $H \in |\mathcal{L}|$  be a general member. In what follows, we repeatedly use the fact that  $\mathcal{H}^k(\mathcal{K}|_H) = \mathcal{H}^k(\mathcal{K})|_H$  for any  $k \in \mathbb{Z}$ , since  $H$  is a general member. We have the distinguished triangle

$$\mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{L} \rightarrow \mathcal{K}|_H \otimes \mathcal{L}.$$

Applying  $\mathbb{R}f_*$  and using (31), we obtain  $\mathbb{R}f_*(\mathcal{K}|_H \otimes \mathcal{L}) = 0$ . Since  $f|_H: H \rightarrow f(H)$  has at most one-dimensional fibers, we have (cf. [3, Lemma 3.1])

$$\mathbb{R}f_*(\mathcal{H}^k(\mathcal{K}|_H \otimes \mathcal{L})) = 0 \quad (32)$$

for any  $k$ . Similarly, applying  $\mathbb{R}f_*$  to the triangle

$$\mathcal{H}^k(\mathcal{K}) \rightarrow \mathcal{H}^k(\mathcal{K} \otimes \mathcal{L}) \rightarrow \mathcal{H}^k(\mathcal{K}|_H \otimes \mathcal{L})$$

and using (32), we obtain

$$\mathbb{R}f_*(\mathcal{H}^k(\mathcal{K})) \cong \mathbb{R}f_*(\mathcal{H}^k(\mathcal{K} \otimes \mathcal{L})). \quad (33)$$

Next let us consider the spectral sequence:

$$E_2^{p,q} = \mathbb{R}^p f_*(\mathcal{H}^q(\mathcal{K})) \Rightarrow \mathbb{R}^{p+q} f_* \mathcal{K}.$$

Since  $E_2^{p,q} = 0$  unless  $0 \leq p \leq 2$ , the above spectral sequence and (31) imply

$$\mathbb{R}^1 f_*(\mathcal{H}^k(\mathcal{K})) = 0, \quad f_*(\mathcal{H}^{k+1}(\mathcal{K})) \cong \mathbb{R}^2 f_*(\mathcal{H}^k(\mathcal{K})) \quad (34)$$

for any  $k$ . By (33) and (34), if we show  $\mathbb{R}^2 f_*(\mathcal{H}^k(\mathcal{K})) = 0$  for any  $k$ , then the conclusion of Assumption 4.8 follows.

Suppose that  $\mathbb{R}^2 f_*(\mathcal{H}^k(\mathcal{K})) \neq 0$  for some  $k$ . By the formal function theorem, there is a closed sub-scheme  $E \subset X$  supported by a two-dimensional fiber of  $f$ , such that  $H^2(E, \mathcal{H}^k(\mathcal{K})|_E) \neq 0$ . By the Grothendieck duality, we have

$$0 \neq H^2(E, \mathcal{H}^k(\mathcal{K})|_E) \cong \text{Hom}_E(\mathcal{H}^k(\mathcal{K})|_E, \mathcal{H}^{-2}(D_E))^\vee.$$

Let  $u: \mathcal{H}^k(\mathcal{K})|_E \rightarrow \mathcal{H}^{-2}(D_E)$  be a non-zero morphism, and consider its image  $\text{Im } u \subset \mathcal{H}^{-2}(D_E)$ . Then the support of  $\text{Im } u$  is two-dimensional because

$$0 \neq \text{Hom}_E(\text{Im } u, \mathcal{H}^{-2}(D_E)) \cong H^2(E, \text{Im } u)^\vee$$

by the duality. Hence by the choice of  $H \in |\mathcal{L}|$ , we may assume that  $(\text{Im } u)|_H \neq 0$ . We may also assume that  $H$  does not contain any associated prime of  $\text{Coker } u$ . Then we can show that  $u|_H: \mathcal{H}^k(\mathcal{K})|_{E \cap H} \rightarrow \mathcal{H}^{-2}(D_E)|_H$  is a non-zero morphism. By adjunction, we have

$$D_{H \cap E} \cong (D_E[-1] \otimes \mathcal{L})|_H.$$

Hence  $u|_H$  induces the non-zero morphism in

$$\text{Hom}_{E \cap H}(\mathcal{H}^k(\mathcal{K})|_{E \cap H}, \mathcal{H}^{-1}(D_{H \cap E} \otimes \mathcal{L}^{-1})).$$

Then the duality on  $E \cap H$  implies

$$0 \neq H^1(E \cap H, \mathcal{H}^k(\mathcal{K})|_{E \cap H} \otimes \mathcal{L}) \cong H^1(E \cap H, \mathcal{H}^k(\mathcal{K}|_H \otimes \mathcal{L})|_E). \quad (35)$$

On the other hand, the surjection

$$\mathcal{H}^k(\mathcal{K}|_H \otimes \mathcal{L}) \twoheadrightarrow \mathcal{H}^k(\mathcal{K}|_H \otimes \mathcal{L})|_E$$

induces the surjection

$$\mathbb{R}^1 f_*(\mathcal{H}^k(\mathcal{K}|_H \otimes \mathcal{L})) \twoheadrightarrow \mathbb{R}^1 f_*(\mathcal{H}^k(\mathcal{K}|_H \otimes \mathcal{L})|_E).$$

However this contradicts (32) and (35), hence it follows that  $\mathbb{R}^2 f_*(\mathcal{H}^k(\mathcal{K})) = 0$ .  $\square$

## 6.2 Crepant resolutions of three dimensional canonical singularities

Let  $0 \in Y = \operatorname{Spec} R$  be a 3-dimensional canonical singularity and  $R$  be a Noetherian complete local ring. Suppose that there is a crepant resolution  $f: X \rightarrow Y$  such that the exceptional locus is an irreducible divisor  $E \subset X$  and  $\mathbb{R}f_*\mathcal{O}_X = \mathcal{O}_Y$ . Then,  $E$  is a generalized del Pezzo surface: that is,  $\omega_E^{-1}$  is ample. We aim to construct a tilting generator of  $D^-(X)$ .

**Lemma 6.2.** *If there is an ample, globally generated line bundle  $\mathcal{L}_1$  on  $E$  with  $H^2(E, \mathcal{L}_1^{-1}) = 0$ , then we have an ample, globally generated line bundle  $\mathcal{L}$  on  $X$  such that  $\mathbb{R}^2 f_*\mathcal{L}^{-1} = 0$ .*

*Proof.* Let  $I_E \subset \mathcal{O}_X$  be the defining ideal of  $E$  and  $E_n \subset X$  the subscheme defined by  $I_E^n$  for  $n > 0$ . Then the obstruction to extend a line bundle  $\mathcal{L}_n \in \operatorname{Pic}(E_n)$  to a line bundle  $\mathcal{L}_{n+1} \in \operatorname{Pic}(E_{n+1})$  lies in  $H^2(E, I_E^n/I_E^{n+1})$ . We have

$$\begin{aligned} H^2(E, I_E^n/I_E^{n+1}) &\cong H^2(E, \mathcal{O}_E(-nE)) \\ &\cong H^0(E, \mathcal{O}_E((n+1)E))^\vee \\ &= 0. \end{aligned}$$

Here, the second isomorphism follows from the Serre duality, and the last isomorphism holds because  $-E$  is  $f$ -ample. Hence, for a given line bundle  $\mathcal{L}_1 \in \operatorname{Pic}(E)$ , we obtain an element

$$\hat{\mathcal{L}} = \{\mathcal{L}_n\}_{n \geq 1} \in \varprojlim \operatorname{Pic}(E_n) \cong \operatorname{Pic}(\hat{X}).$$

By the Grothendieck existence theorem, there is a line bundle  $\mathcal{L}$  on  $X$  such that  $\mathcal{L}|_{\hat{X}} \cong \hat{\mathcal{L}}$ .

Take an ample, globally generated line bundle  $\mathcal{L}_1$  on  $E$  such that  $H^2(E, \mathcal{L}_1^{-1}) = 0$ . Let  $\mathcal{L} \in \text{Pic}(X)$  be its extension. We have

$$\begin{aligned} H^2(E, \mathcal{L}_1^{-1} \otimes I_E^n / I_E^{n+1}) &\cong H^0(E, \mathcal{L}_1 \otimes \mathcal{O}_E(nE) \otimes \omega_E)^\vee \\ &= 0, \end{aligned}$$

since  $-E$  is  $f$ -ample and  $H^0(E, \mathcal{L}_1 \otimes \omega_E) = 0$ . Hence  $\mathbb{R}^2 f_* \mathcal{L}^{-1} = 0$  by the formal function theorem.  $\mathcal{L}$  is also globally generated by the basepoint free theorem, and clearly  $\mathcal{L}$  is ample.  $\square$

In particular, Theorem 6.1 implies the following.

**Theorem 6.3.** *In the situation of Lemma 6.2, there is a tilting generator of  $D^-(X)$ .*

**Example 6.4.** There is a 3-dimensional crepant resolution  $f: X \rightarrow Y$  from a Calabi-Yau threefold  $X$  defined over  $\mathbb{C}$  whose exceptional locus is isomorphic to  $E$  in (i), (ii) below ([11], [12]). Replace  $Y$  with its completion at the singular point and shrink  $X$  accordingly.

We show the existence of tilting generators of  $D^-(X)$ . The key fact is that if we have a line bundle  $\mathcal{L}_1$  on  $E$ , as in Lemma 6.2, then we can find a tilting generator of  $D^-(X)$  by Theorem 6.3.

- (i) The first example is a quadric  $E \subset \mathbb{P}^3$ , that is,  $E$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  or the cone over a conic. Then  $\mathcal{L}_1 = \mathcal{O}_{\mathbb{P}^3}(1)|_E$  satisfies  $H^2(E, \mathcal{L}_1^{-1}) = 0$ .
- (ii) For the second example, take the cone over a conic  $S \subset \mathbb{P}^3$ . Let  $E$  be a surface obtained by the blowing-up  $\pi: E \rightarrow S$  at a non-singular point in  $S$ . Note that  $E$  is a singular del Pezzo surface. Denote by  $C$  the exceptional curve of  $\pi$  and put  $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^3}(1)|_S$ . Then  $\mathcal{L}_1 = \pi^* \mathcal{O}(1) \otimes \mathcal{O}_E(-C)$  is an ample, globally generated line bundle satisfying  $H^2(E, \mathcal{L}_1^{-1}) = 0$ .

## 7 The cotangent bundle of $G(2, 4)$

In §7.1, we cite and prove some results that §7.2 uses. In §7.2, we find tilting generators on a one-parameter deformation of the cotangent bundle  $X_0 = T^*G(2, 4)$ , where  $G(2, 4)$  is the Grassmann manifold. We assume all varieties are defined over  $\mathbb{C}$  in this section.

### 7.1 The Bott theorem

Let  $G$  be the Grassmann manifold  $G(k, V)$  of  $k$ -dimensional subspaces in an  $n$ -dimensional  $\mathbb{C}$ -vector space  $V$ . There is a non-split exact sequence

$$0 \rightarrow \Omega_G \rightarrow \tilde{\Omega}_G \rightarrow \mathcal{O}_G \rightarrow 0$$

corresponding to a nonzero element of the 1-dimensional space  $H^1(G, \Omega_G)$ . Put  $\tilde{T}_G = (\tilde{\Omega}_G)^\vee$ . We denote the total space of  $\tilde{\Omega}_G$  (resp.  $\Omega_G$ ) by  $X$  (resp.  $X_0$ ). Then there is a one-parameter deformation ([17], [14])

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \downarrow \\ & & \mathbb{A}^1 \end{array}$$

of the Springer resolution

$$f_0: X_0 \rightarrow Y_0 = \text{Spec } R_0.$$

We denote by  $\pi: X \rightarrow G$  and  $\pi_0: X_0 \rightarrow G$  the projections.

Let  $\mathcal{U}$  be the tautological  $k$ -dimensional sub-bundle of  $\mathcal{O}_G \otimes V$ . We also define  $\mathcal{U}^\perp$  to be  $((\mathcal{O}_G \otimes V)/\mathcal{U})^\vee$ , the dual of the quotient bundle. For a vector bundle  $\mathcal{E}$  of rank  $m$  on  $G$ , we consider the associated principal  $GL(m, \mathbb{C})$ -bundle and denote by  $\Sigma^\alpha \mathcal{E}$  the vector bundle associated with the  $GL(m, \mathbb{C})$  representation of highest weight  $\alpha \in \mathbb{Z}^m$ . For  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m$  with  $\alpha_1 \geq \dots \geq \alpha_m$  (such a sequence is called a *non-increasing* sequence), we have

$$\Sigma^\alpha(\mathcal{E}^\vee) = \Sigma^{(-\alpha_m, \dots, -\alpha_1)} \mathcal{E} = (\Sigma^\alpha \mathcal{E})^\vee.$$

We have the following equality:

$$\begin{aligned} & \text{Hom}_G^i(\Sigma^\alpha \mathcal{U}, \Sigma^\beta \mathcal{U} \otimes (\bigoplus_{n \geq 0} \text{Sym}^n(T_G))) \\ &= \bigoplus_{n \geq 0} \bigoplus_{|\lambda|=n} H^i(G, \Sigma^\alpha \mathcal{U}^\vee \otimes \Sigma^\beta \mathcal{U} \otimes \Sigma^\lambda \mathcal{U}^\vee \otimes \Sigma^\lambda (\mathcal{U}^\perp)^\vee). \end{aligned} \quad (36)$$

Here  $|\lambda| = \sum \lambda_i$  and all the  $\lambda_i$ 's are non-negative. For the proof of (36), see [5, page 80] and use  $T_G = \mathcal{U}^\vee \otimes (\mathcal{U}^\perp)^\vee$ .

**Lemma 7.1.** *Suppose that the vector space in (36) is 0-dimensional for fixed  $i$ ,  $\alpha$  and  $\beta$ . Then the vector space  $\text{Hom}_X^i(\pi^* \Sigma^\alpha \mathcal{U}, \pi^* \Sigma^\beta \mathcal{U})$  is also 0-dimensional.*

*Proof.* The assertion follows from the equality

$$\begin{aligned} \text{Hom}_X^i(\pi^* \Sigma^\alpha \mathcal{U}, \pi^* \Sigma^\beta \mathcal{U}) &\cong \text{Hom}_G^i(\Sigma^\alpha \mathcal{U}, \Sigma^\beta \mathcal{U} \otimes \pi_* \mathcal{O}_X) \\ &\cong \text{Hom}_G^i(\Sigma^\alpha \mathcal{U}, \Sigma^\beta \mathcal{U} \otimes (\bigoplus_{n \geq 0} \text{Sym}^n(\tilde{T}_G))) \end{aligned}$$

and the filtration

$$\text{Sym}^n(\tilde{T}_G) = F^0 \supset F^1 \supset \dots \supset F^n \supset F^{n+1} = 0$$

with  $F^l/F^{l+1} \cong \text{Sym}^{n-l}(T_G)$ . □

Let  $F(V)$  be the flag variety of  $GL(V)$  and

$$\mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots \subset \mathcal{U}_{n-1} \subset \mathcal{U}_n = V \otimes \mathcal{O}_{F(V)}.$$

the sequence of the universal sub-bundles  $\mathcal{U}_i$  of rank  $i$ . We put

$$\mathcal{O}(\delta_1, \dots, \delta_n) = \mathcal{U}_1^{-\delta_1} \otimes (\mathcal{U}_2/\mathcal{U}_1)^{-\delta_2} \otimes \cdots \otimes (\mathcal{U}_n/\mathcal{U}_{n-1})^{-\delta_n}.$$

The following lemma is taken from the proof of [9, Proposition 2.2].

**Lemma 7.2.** *For non-increasing sequences  $\alpha \in \mathbb{Z}^k, \beta \in \mathbb{Z}^{n-k}$ , we have*

$$H^i(G, \Sigma^\alpha \mathcal{U}^\vee \otimes \Sigma^\beta \mathcal{U}^\perp) = H^i(F(V), \mathcal{O}(\Delta)),$$

where  $\Delta = (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{n-k})$ .

By Lemma 7.1 and Lemma 7.2, showing the vanishing of the vector space

$$\mathrm{Hom}_X^i(\pi^* \Sigma^\alpha \mathcal{U}, \pi^* \Sigma^\beta \mathcal{U})$$

is reduced to the dimension counting of the cohomology  $H^i(F(V), \mathcal{O}(\Delta))$  on the flag variety  $F(V)$ . Hence, we shall compute  $H^i(F(V), \mathcal{O}(\Delta))$  for  $\Delta = (\delta_1, \dots, \delta_n) \in \mathbb{Z}^n$ . The permutation group  $\mathfrak{S}_n$  naturally acts on  $\mathbb{Z}^n$ :

$$\sigma(\delta_1, \dots, \delta_n) = (\delta_{\sigma(1)}, \dots, \delta_{\sigma(n)}).$$

We also define the tilde action of  $\mathfrak{S}_n$  on  $\mathbb{Z}^n$ :

$$\tilde{\sigma}(\Delta) = \sigma(\Delta + \rho) - \rho.$$

Here  $\rho = (n-1, n-2, \dots, 0)$ . For instance, when we put  $\sigma_l = (l \ l+1)$ , we obtain

$$\tilde{\sigma}_l(\delta_1, \dots, \delta_n) = (\delta_1, \dots, \delta_{l-1}, \delta_{l+1} - 1, \delta_l + 1, \delta_{l+2}, \dots, \delta_n).$$

The Bott theorem implies that:

(1) If  $\Delta$  is non-increasing, then we have

$$H^i(F(V), \mathcal{O}(\Delta)) = \begin{cases} \Sigma^\Delta V & i = 0 \\ 0 & i > 0. \end{cases}$$

(2) If  $\Delta$  is not non-increasing, then we apply the tilde action of  $\mathfrak{S}_n$  for transpositions like  $\sigma_l = (l \ l+1)$ , trying to move bigger numbers to the right past smaller numbers. Repeat this process. Then there are two possibilities:

- Suppose that eventually, we achieve  $\delta_{l+1} = \delta_l + 1$  for some  $l$ . Then  $H^i(F(V), \mathcal{O}(\Delta)) = 0$  for all  $i$ .
- Suppose that after applying  $j$  times tilde actions of transpositions in  $\mathfrak{S}_n$ , we can transform  $\Delta$  into a non-increasing sequence  $\Delta_0$ . Then we have

$$H^i(F(V), \mathcal{O}(\Delta)) = \begin{cases} \Sigma^{\Delta_0} V & i = j \\ 0 & i \neq j. \end{cases}$$

## 7.2 $G(2, 4)$

Henceforth in this section,  $G$  denotes  $G(2, 4)$ . Let us find a tilting generator of  $D^-(X)$  using Theorem 4.16 in this subsection. Let  $\mathcal{O}_G(1) = \Sigma^{(-1, -1)}\mathcal{U}$  be a line bundle on  $G$  which gives the Plücker embedding  $G \hookrightarrow \mathbb{P}^5$  and we denote  $\pi^*\mathcal{O}_G(1)$  by  $\mathcal{O}_X(1)$ .

First we want to show (3) for  $\mathcal{L} = \mathcal{O}_X(1)$ ; namely

$$H^i(X, \mathcal{O}_X(-j)) (= \text{Hom}_X^i(\pi^*\Sigma^{(0,0)}\mathcal{U}, \pi^*\Sigma^{(j,j)}\mathcal{U})) = 0 \quad (37)$$

for  $0 < j < 4$  and  $i \geq 2$ . Putting  $\alpha = (0, 0)$  and  $\beta = (j, j)$  in (36) and using Lemma 7.2, we obtain

$$\begin{aligned} & \text{Hom}_G^i(\Sigma^{(0,0)}\mathcal{U}, \Sigma^{(j,j)}\mathcal{U} \otimes (\bigoplus_{n \geq 0} \text{Sym}^n(T_G))) \\ &= \bigoplus_{n \geq 0} \bigoplus_{|\lambda|=n} H^i(G, \Sigma^{(j-\lambda_2, j-\lambda_1)}\mathcal{U} \otimes \Sigma^{(-\lambda_2, -\lambda_1)}\mathcal{U}^\perp) \\ &= \bigoplus_{n \geq 0} \bigoplus_{|\lambda|=n} H^i(F(V), \mathcal{O}(\lambda_1 - j, \lambda_2 - j, -\lambda_2, -\lambda_1)), \end{aligned} \quad (38)$$

where we put  $\lambda = (\lambda_1, \lambda_2)$ . For the proof of (37), by Lemma 7.1, it is enough to see the vanishing of (38) for  $0 < j < 4$  and  $i \geq 2$ .

Denote

$$\Delta = (\lambda_1 - j, \lambda_2 - j, -\lambda_2, -\lambda_1)$$

below. The Bott theorem says that one of the following occurs:

- If  $\lambda_2 - j \geq -\lambda_2$ , then  $H^i(F(V), \mathcal{O}(\Delta)) = 0$  if and only if  $i \neq 0$ .
- If  $\lambda_2 - j + 1 = -\lambda_2$ , then  $H^i(F(V), \mathcal{O}(\Delta)) = 0$  for all  $i$ .
- If  $\lambda_2 - j + 1 < -\lambda_2$ , then  $\lambda_2 = 0$  and  $j = 2, 3$ , which implies

$$\tilde{\sigma}_2\Delta = (\lambda_1 - j, -1, -j + 1, -\lambda_1).$$

In the case  $\lambda_1 - j \geq -1$ ,  $H^i(F(V), \mathcal{O}(\Delta)) \neq 0$  implies  $i = 1$ . In the case  $\lambda_1 - j + 1 = -1$ ,  $H^i(F(V), \mathcal{O}(\Delta)) = 0$  for all  $i$ . In the case  $\lambda_1 - j + 1 < -1$ , we obtain  $\lambda_1 = 0$  and  $j = 3$ . Then it is easy to see that  $H^i(F(V), \mathcal{O}(\Delta)) = 0$  for all  $i$ .

Therefore we obtain (37) as desired.

Next we want to check that Assumption 4.8 is true, i.e.  $\mathcal{K} \in D(X)$  satisfies the equality

$$\mathbb{R}f_*(\mathcal{H}^k(\mathcal{K}) \otimes \mathcal{O}_X(j)) = 0 \quad (39)$$

for any  $k$  and  $j$ , ( $0 \leq j \leq 3$ ) when we assume the equalities

$$\mathbb{R}f_*(\mathcal{K} \otimes \mathcal{O}_X(j)) = 0 \quad (40)$$



for any  $j$ , ( $0 \leq j \leq 3$ ). Because  $\mathcal{O}_X(1)$  gives an embedding  $h: X \hookrightarrow \mathbb{P}_R^5$ , we can say that (40) is equivalent to

$$\mathbb{R}g_*(h_*\mathcal{K} \otimes \mathcal{O}(j)) = 0 \quad (41)$$

for all  $j$  with  $0 \leq j \leq 3$ , where  $g: \mathbb{P}_R^5 \rightarrow \text{Spec } R$  is the structure morphism. On the other hand,  $D(\mathbb{P}_R^5)$  has a semi-orthogonal decomposition

$$D(\mathbb{P}_R^5) = \left\langle g^*D(R) \otimes \mathcal{O}_{\mathbb{P}_R^5}(-5), g^*D(R) \otimes \mathcal{O}_{\mathbb{P}_R^5}(-4), \dots, g^*D(R) \right\rangle,$$

and hence it follows from our assumption (41) that

$$h_*\mathcal{K} \in \left\langle g^*D(R) \otimes \mathcal{O}_{\mathbb{P}_R^5}(-5), g^*D(R) \otimes \mathcal{O}_{\mathbb{P}_R^5}(-4) \right\rangle.$$

Consequently, there is a triangle

$$\cdots \rightarrow g^*W_{-4} \otimes_R \mathcal{O}_{\mathbb{P}_R^5}(-4) \rightarrow h_*\mathcal{K} \rightarrow g^*W_{-5} \otimes_R \mathcal{O}_{\mathbb{P}_R^5}(-5) \rightarrow \cdots$$

for some  $W_i \in D(R)$ , and then we obtain a long exact sequence

$$\cdots \rightarrow \mathcal{H}^k(W_{-4}) \otimes_R \mathcal{O}_{\mathbb{P}_R^5}(-4) \rightarrow \mathcal{H}^k(h_*\mathcal{K}) \rightarrow \mathcal{H}^k(W_{-5}) \otimes_R \mathcal{O}_{\mathbb{P}_R^5}(-5) \rightarrow \cdots.$$

Because the support of  $\mathcal{H}^k(h_*\mathcal{K})$  is contained in  $X$  and the support of  $\mathcal{H}^k(W_{-5}) \otimes_R \mathcal{O}_{\mathbb{P}_R^5}(-5)$  is the inverse image of some closed subset on  $Y$  by  $g$ , the morphism  $\mathcal{H}^k(h_*\mathcal{K}) \rightarrow \mathcal{H}^k(W_{-5}) \otimes_R \mathcal{O}_{\mathbb{P}_R^5}(-5)$  should be zero. Therefore we have a short exact sequence

$$0 \rightarrow \mathcal{H}^{k-1}(W_{-5}) \otimes_R \mathcal{O}_{\mathbb{P}_R^5}(-5) \rightarrow \mathcal{H}^k(W_{-4}) \otimes_R \mathcal{O}_{\mathbb{P}_R^5}(-4) \rightarrow \mathcal{H}^k(h_*\mathcal{K}) \rightarrow 0.$$

Then (39) follows. Now we can construct a tilting generator of  $D^-(X)$  by Theorem 4.16.

We have proved the following:

**Theorem 7.3.** *The derived category  $D^-(X)$  has a tilting generator which is a vector bundle on  $X$ .*

**Corollary 7.4** (cf. [8]). *The derived category  $D^-(X_0)$  has a tilting generator which is a vector bundle on  $X_0$ .*

*Proof.* Let  $\mathcal{E}$  be a tilting generator in  $D^-(X)$  constructed above. Put  $\mathcal{E}_0 = i^*\mathcal{E}$ , where  $i: X_0 \hookrightarrow X$  is the embedding. Since  $X$  is a one-parameter deformation of  $X_0$ , there is an exact sequence  $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_0} \rightarrow 0$ . Taking a tensor product with  $\mathcal{E}$ , we obtain an exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{E} \rightarrow \mathcal{E}_0 \rightarrow 0. \quad (42)$$

Applying  $\mathbb{R}\text{Hom}(\mathcal{E}, -)$  to (42), we can conclude that  $\mathcal{E}_0$  is a tilting object. We can directly check that  $\mathcal{E}_0$  is a generator.  $\square$

## 8 Auxiliary result: the existence of a right adjoint functor

In this section, we show the existence of a right adjoint functor of  $\Phi_{n-1}$ , which is needed in §4.4. Let  $Y$  be a scheme of finite type over a field or a spectrum of a Noetherian complete local ring. This condition assures the existence of the dualizing complex on  $Y$ . Let us consider a projective morphism between schemes  $f: X \rightarrow Y$ . Then we know that  $R = H^0(X, \mathcal{O}_X)$  has the dualizing complex  $D_R$ . For a vector bundle  $\mathcal{E}$  on  $X$ , put

$$\begin{aligned} \mathcal{A} &= \mathcal{E}nd_X \mathcal{E}, \quad A = \text{End}_X \mathcal{E}, \\ D_{\mathcal{A}} &= \mathbb{R}\mathcal{H}om_X(\mathcal{A}, D_X), \quad D_A = \mathbb{R}\text{Hom}_R(A, D_R), \\ \mathbb{D}_{\mathcal{A}}(-) &= \mathbb{R}\mathcal{H}om_{\mathcal{A}}(-, D_{\mathcal{A}}): D^-(\mathcal{A}) \rightarrow D^+(\mathcal{A}^\circ), \\ \mathbb{D}_A(-) &= \mathbb{R}\text{Hom}_A(-, D_A): D^-(A) \rightarrow D^+(A^\circ), \\ \tilde{\Phi}(-) &= \mathbb{R}\mathcal{H}om_X(\mathcal{E}, -): D^-(X) \rightarrow D^-(\mathcal{A}), \\ \Phi(-) &= \mathbb{R}\text{Hom}_X(\mathcal{E}, -): D^-(X) \rightarrow D^-(A), \\ \Psi(-) &= (-) \otimes_A^{\mathbb{L}} \mathcal{E}: D^-(A) \rightarrow D^-(X), \\ \mathbb{D}_R &= \mathbb{R}\text{Hom}_R(-, D_R): D^-(R) \rightarrow D^+(R). \end{aligned}$$

For the dual vector bundle  $\mathcal{E}^\vee$  of  $\mathcal{E}$ , we put

$$\tilde{\Phi}^\circ = \mathbb{R}\mathcal{H}om_X(\mathcal{E}^\vee, -): D^+(X) \rightarrow D^+(\mathcal{A}^\circ).$$

Lemma 8.1 must be well-known to specialists. When  $\mathcal{E} = \mathcal{O}_X$ , the lemma is a paraphrase of the Grothendieck duality for the natural projective morphism  $g: X \rightarrow \text{Spec } R$ .

**Lemma 8.1.**  $\mathbb{D}_A \circ \Phi \cong \Phi \circ \mathbb{D}_X$ .

*Proof.* We have a diagram:

$$\begin{array}{ccccc} D^-(X) & \xrightarrow{\tilde{\Phi}} & D^-(\mathcal{A}) & \xrightarrow{\mathbb{R}\Gamma} & D^-(A) \\ \mathbb{D}_X \downarrow & & \mathbb{D}_{\mathcal{A}} \downarrow & & \mathbb{D}_A \downarrow \\ D^+(X) & \xrightarrow{\tilde{\Phi}^\circ} & D^+(\mathcal{A}^\circ) & \xrightarrow{\mathbb{R}\Gamma} & D^+(A^\circ). \end{array} \quad (43)$$

We note that there is an isomorphism  $\Phi \cong \mathbb{R}\Gamma \circ \tilde{\Phi}$  and that  $\tilde{\Phi}$  gives an equivalence of derived categories ([18]).

First, we show that the left diagram in (43) is commutative. For  $\mathcal{N} \in$

$D^-(X)$ , we have

$$\begin{aligned}
\mathbb{D}_{\mathcal{A}} \circ \tilde{\Phi}(\mathcal{N}) &\cong \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{N}), D_{\mathcal{A}}) \\
&\cong \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{N}), \mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{E} \otimes D_X)) \\
&\cong \mathbb{R}\mathcal{H}om_X(\mathcal{N}, \mathcal{E} \otimes D_X) \\
&\cong \mathbb{R}\mathcal{H}om_X(\mathcal{E}^\vee, \mathbb{R}\mathcal{H}om_X(\mathcal{N}, D_X)) \\
&\cong \tilde{\Phi}^\circ \circ \mathbb{D}_X(\mathcal{N}).
\end{aligned} \tag{44}$$

Here, the isomorphism (44) comes from the Morita equivalence  $\text{Coh } U \cong \text{Coh } \mathcal{A}|_U$  on every affine open set  $U \subset X$ .

Therefore, it remains to show that the right diagram in (43) is commutative. The Grothendieck duality for  $g: X \rightarrow \text{Spec } R$  implies

$$\mathbb{R}\Gamma(D_{\mathcal{A}}) \cong \mathbb{R}\text{Hom}_R(\mathbb{R}\Gamma(\mathcal{A}), D_R).$$

Composing this isomorphism with the natural morphism  $A \rightarrow \mathbb{R}\Gamma(\mathcal{A})$ , we obtain the morphism

$$\mathbb{R}\Gamma(D_{\mathcal{A}}) \rightarrow D_A. \tag{45}$$

Moreover since we have

$$\text{Hom}_A(M, \text{Hom}_R(A, N)) \cong \text{Hom}_R(M, N)$$

for any  $M \in \text{mod } A$ ,  $N \in R\text{mod}$ , we have the isomorphism,

$$\mathbb{R}\text{Hom}_A(M, D_A) \cong \mathbb{R}\text{Hom}_R(M, D_R) \tag{46}$$

in  $D^-(R)$  for  $M \in D^-(A)$ .

For  $\mathcal{M} \in D^-(\mathcal{A})$ , we have the following sequence of isomorphisms and natural morphisms,

$$\begin{aligned}
\mathbb{R}\Gamma \circ \mathbb{D}_{\mathcal{A}}(\mathcal{M}) &= \mathbb{R}\Gamma \circ \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, D_{\mathcal{A}}) \\
&\cong \mathbb{R}\text{Hom}_{\mathcal{A}}(\mathcal{M}, D_{\mathcal{A}}) \\
&\rightarrow \mathbb{R}\text{Hom}_{\mathcal{A}}(\tilde{\Phi} \circ \Psi \circ \mathbb{R}\Gamma(\mathcal{M}), D_{\mathcal{A}})
\end{aligned} \tag{47}$$

$$\cong \mathbb{R}\text{Hom}_{\mathcal{A}}(\mathbb{R}\Gamma(\mathcal{M}), \mathbb{R}\Gamma(D_{\mathcal{A}})) \tag{48}$$

$$\rightarrow \mathbb{R}\text{Hom}_A(\mathbb{R}\Gamma(\mathcal{M}), D_A) \tag{49}$$

$$= \mathbb{D}_A \circ \mathbb{R}\Gamma(\mathcal{M}).$$

Here the morphism (47) and the isomorphism (48) are obtained from the fact that  $\tilde{\Phi} \circ \Psi$  is a left adjoint functor of  $\mathbb{R}\Gamma$ , and moreover the morphism (49) comes from the morphism (45). Consequently we obtain a morphism of functors

$$\phi: \mathbb{R}\Gamma \circ \mathbb{D}_{\mathcal{A}} \rightarrow \mathbb{D}_A \circ \mathbb{R}\Gamma.$$

Next we want to check that  $\phi$  is an isomorphism. Note that it is enough to check that  $\phi$  is isomorphic after applying the forgetful functor  $D^-(A) \rightarrow$

$D^-(R)$ . Take  $\mathcal{N} \in D^-(X)$  such that  $\tilde{\Phi}(\mathcal{N}) = \mathcal{M}$ . Then, because of the commutativity of the left diagram in (43), we have

$$\mathbb{R}\Gamma \circ \mathbb{D}_A(\mathcal{M}) \cong \mathbb{R}\Gamma \circ \mathbb{D}_X(\mathcal{E}^\vee \otimes \mathcal{N}).$$

We also have

$$\begin{aligned} \mathbb{D}_A \circ \mathbb{R}\Gamma(\mathcal{M}) &\cong \mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{N}), D_A) \\ &\cong \mathbb{R}\mathrm{Hom}_R(\mathbb{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{N}), D_R) \\ &\cong \mathbb{D}_R \circ \mathbb{R}\Gamma(\mathcal{E}^\vee \otimes \mathcal{N}). \end{aligned}$$

by (46). Then the Grothendieck duality for  $g$  implies that  $\phi$  is isomorphic.  $\square$

Put

$$D^\dagger(X) = \left\{ \mathcal{K} \in D(X) \mid \Phi(\mathcal{K}) \in D^b(A) \right\}.$$

**Lemma 8.2.** *The functor  $\Phi : D^\dagger(X) \rightarrow D^b(A)$  has a right adjoint functor.*

*Proof.* Indeed, using  $\mathbb{D}_A \circ \Phi \cong \Phi \circ \mathbb{D}_X$ , we can readily check that  $\mathbb{D}_X \circ \Psi \circ \mathbb{D}_A$  is a right adjoint functor of  $\Phi$ .  $\square$

## A Non-commutative crepant resolution

First, let us recall the definition of non-commutative crepant resolutions introduced by Van den Bergh [19].

**Definition A.1.** Let  $k$  be a field,  $R$  a normal Gorenstein finitely generated  $k$ -domain. Furthermore we denote by  $A$  an  $R$ -algebra that is finitely generated as an  $R$ -module.  $A$  is called a non-commutative crepant resolution of  $R$  if the following conditions hold:

- (i) There is a reflexive  $R$ -module  $E$  such that  $A = \mathrm{End}_R(E)$ .
- (ii) The global dimension of  $A$  is finite.
- (iii)  $A$  is a Cohen-Macaulay  $R$ -module.

The next assertion is essentially shown in [19].

**Proposition A.2.** *Let  $Y = \mathrm{Spec} R$  be an affine normal Gorenstein variety and assume that there is a crepant resolution  $f : X \rightarrow Y$ : that is,  $f$  is a birational projective morphism from a smooth variety  $X$  and  $f^*\omega_Y = \omega_X$ . If we have a tilting generator  $\mathcal{E}$  of  $D^-(X)$  such that*

$$\mathrm{Hom}_X^i(\mathcal{E}, \mathcal{O}_X) = \mathrm{Hom}_X^i(\mathcal{O}_X, \mathcal{E}) = 0$$

*for  $i \neq 0$ , then  $R$  has a non-commutative crepant resolution.*

*Proof.* When  $\dim R \leq 1$ , then  $R$  is itself a non-commutative resolution of  $R$ . Thus, we assume that  $\dim R \geq 2$  in what follows. We define as

$$\begin{aligned} E &= \mathbb{R}\Gamma(\mathcal{E})(\cong \mathbb{R}^0\Gamma(\mathcal{E})), \quad \mathcal{A} = \mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{E}), \\ A &= \mathbb{R}\Gamma(\mathcal{A})(\cong \mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{E}) \cong \mathcal{H}om_X(\mathcal{E}, \mathcal{E})). \end{aligned}$$

By  $f^*\omega_Y = \omega_X$ , we have  $f^!\mathcal{O}_Y = \mathcal{O}_X$ . Then the Grothendieck duality and our assumptions imply that

$$\begin{aligned} \mathrm{Hom}_R^i(E, R) &\cong \mathrm{Hom}_X^i(\mathcal{E}, f^!\mathcal{O}_Y) \\ &\cong \mathrm{Hom}_X^i(\mathcal{E}, \mathcal{O}_X) \\ &= 0 \end{aligned}$$

for any  $i \neq 0$ , which implies that  $E$  is Cohen-Macaulay. We can show similarly that  $A$  is Cohen-Macaulay, since

$$\begin{aligned} \mathrm{Hom}_X^i(\mathcal{A}, \mathcal{O}_X) &\cong \mathrm{Hom}_X^i(\mathcal{O}_X, \mathcal{A}) \\ &\cong \mathrm{Hom}_X^i(\mathcal{E}, \mathcal{E}) = 0 \end{aligned}$$

for any  $i \neq 0$ . Note that  $\mathrm{End}_R(E)$  and  $A$  are reflexive, since they are Cohen-Macaulay and  $\dim R \geq 2$ . Then the natural homomorphism  $A \rightarrow \mathrm{End}_R(E)$  is isomorphic in codimension one, as well as everywhere else. Moreover,  $D^b(A)$  and  $D^b(X)$  are derived equivalent, and therefore the global dimension of  $A$  is finite.  $\square$

**Corollary A.3.** *Let  $Y = \mathrm{Spec} R$  be an affine normal Gorenstein variety defined over  $\mathbb{C}$ , and suppose that there is a crepant resolution  $f: X \rightarrow Y$  with at most two-dimensional fibers. Further assume that we have a globally generated, ample line bundle  $\mathcal{L}$  on  $X$  which satisfies  $\mathbb{R}^2 f_* \mathcal{L}^{-1} = 0$ . Then  $R$  has a non-commutative crepant resolution.*

*Proof.* Note that  $\mathbb{R}f_* \mathcal{O}_X \cong \mathcal{O}_Y$  by the vanishing theorem. Because  $\mathcal{O}_X$  is a direct summand of the tilting generator  $\mathcal{E}$  constructed in Theorem 6.1 we obtain

$$\mathrm{Hom}_X^i(\mathcal{E}, \mathcal{O}_X) = \mathrm{Hom}_X^i(\mathcal{O}_X, \mathcal{E}) = 0$$

for  $i \neq 0$ . We can apply the above proposition.  $\square$

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